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Minimax Regret filter for uncertainty Single-Input Single-Output systems: simulation study

Filtro de arrepentimiento minimax para sistemas de única entrada y salida inciertos: estudio de simulación

José Perea-Arango [1*], Piotr Graczyk [2], Juan Pablo Fernández-Gutiérrez [3]

[1]Departamento de Negocios, Empresas Públicas de Medellín EPM. Carrera 58 42-125. C. P. 050015. Colombia, Medellín
[2]LAREMA, Université d'Angers. Présidence 40, rue de Rennes. BP 73532.49035. France, Angers
[3]Facultad de Ciencias Básicas, Universidad de Medellín. Carrera 87 30-65. C. P. 050026. Colombia, Medellín

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ABSTRACT: The Kalman filter, widely used since its introduction in 1960, assumes Gaussian random disturbances. However, this assumption can be inappropriate in non-Gaussian contexts, leading to suboptimal performance. Researchers have proposed robust filters like minimax filters to address this limitation, but these filters can overly conservative estimates. This research introduces a novel approach that combines unknown-but-bounded dynamics for the state process and stochastic processes for the measurement equation along with a Minimax Regret framework to improve state estimation in one-dimensional linear dynamic models. We evaluate the proposed method through two simulation studies. The first study optimizes the hyperparameter value using Grid Search. In contrast, the second compares the performance of the proposed method with conventional methods, including the Kalman filter and a robust version of the RobKF filter implemented in R software, using a suitable performance metric such as mean squared error. The results demonstrate the superiority of the proposed algorithm.

RESUMEN: El filtro de Kalman, ampliamente utilizado desde su introducción en 1960, asume ruidos aleatorios gaussianos. Sin embargo, este supuesto puede ser inapropiado en contextos cuyas variables no provienen de una distribución normal, lo que lleva a un desempeño subóptimo del filtro. Los investigadores han propuesto filtros robustos como el filtro minimax para abordar esta limitación, pero estos pueden proporcionar estimaciones demasiado conservadoras. Este trabajo presenta un enfoque novedoso que combina dinámicas desconocidas pero acotadas para el proceso de estado y procesos estocásticos para la ecuación de medidas, junto con un marco de Arrepentimiento Minimax para mejorar la estimación del estado en modelos dinámicos lineales de una dimensión. Evaluamos el método propuesto a través de dos estudios de simulación. El primer estudio utiliza el algoritmo de búsqueda por malla para optimizar el valor del hiperparámetro, mientras que el segundo estudia el desempeño del método propuesto en comparación con métodos convencionales, incluyendo el filtro de Kalman y una versión robusta del filtro RobKF implementada en software R. Los resultados demuestran la superioridad de nuestro algoritmo propuesto.

1. Introduction

The Equations (1a) - (1b) were proposed by [1] to represent the uncertain linear system with multiple-input multiple-output setting

$$x_{t+1} = A_t x_t + \xi_t \tag{1a}$$

$$y_t = H_t x_t + w_t,\ t = 1, 2, \dots \tag{1b}$$

where the estimation of the state process x_t is given solving the problem [2]

$$\widetilde{x}_{t|t} := \arg\min_{\widetilde{x}\sim\mathcal{F}_t} E\left\{ |x_t - \widetilde{x}|^2 \right\} \tag{2}$$

* Corresponding author: José Perea-Arango
E-mail: jose.perea@epm.com.co
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where $\mathcal{F}_t$ is the σ-algebra generated by $\{y_1, \dots y_t\}$. This can be reduced for the case Single-Input Single-Output (SISO) processes with uncertainty. Since then, for the convenient mathematical tractability of the assumption of Gaussian disturbances of the processes (1a) - (1b), many extensions have been proposed. The assumption of Gaussian disturbances is inappropriate in several contexts. The objective function of (2) is known as the Mean Squared Error (MSE), and changing its form allows new concepts of filtering to be built.

Several versions of the robust Kalman Filter (KF) have been applied to many areas, spanning from aerospace technology to control engineering and robotics [2–5], and they can also be studied in the context of various state forecasting problems. The most applied approach has been the minimax method [6–11]. Among them, H_2 and H_∞ are two classical approaches, but they have a few drawbacks. In the H_2 setting, the stochastic assumptions are not easy to verify and under the assumption of normality, this has shown to have a lack of robustness to outliers data with the consequence of predicting inaccurate future states [12]. Regarding the minimax approach, although it has been widely used as a robust method in the estimation of states x_t, its performance is often unsatisfactory, since it tends to be overly conservative [13].

The theory of linear estimation in Krein Spaces is based on simple concepts such as projections and matrix factorizations and leads to an interesting connection between Krein space projection and the recursive computation of the stationary points of certain second-order (or quadratic) forms [14]. Furthermore, they obtain upper and lower bounds for the H_∞ norm of the Kalman filter and the recursive-least-squares (RLS) algorithm, concerning prediction and filter errors. Moreover, the main conclusion in [15] is shown that "the H_∞ for RLS is data dependent, whereas, for least-mean-squares (LMS) algorithms and normalized LMS, the H_∞ is simply unity". Approaches, such as Outlier based t-distribution distribution by [16], infinity variance by [17], and the bounded [18–20], interval [21], and unknown-but-bounded [22, 23] have been applied to filtering.

Instead of considering the Kalman stochastic filter or the minimax approach, this paper approaches state estimation from a novel point of view, where, in lieu of using the classical MSE, as it is in the previous approaches, we use an objective function to be minimized that counterbalances the conservative nature of the minimax approach and does not rely on assumptions of non-measurable states. Consequently, we do not impose any distribution to the state equation. Namely, we propose to use the Minimax Regret concept [24] for the filtering problem, in particular, SISO systems described by univariate dynamical linear systems. Due to its focus on minimizing maximum regret, the minimax regret approach exhibits reduced susceptibility to outliers or extreme prediction errors, thereby alleviating potential biases in the MSE estimator [25]. [26, 27] have seized this vision with different uncertainty sets in the dynamic case, and [13, 28] in the static case. Thus, our article is organized as follows.

In section 2, we present the mathematical model of the new approach to estimate the states of a uni-dimensional linear dynamic system. Section 3 presents the main theorem, where we use different optimization tools to get a closed form of the real-valued gain for our uncertainty linear SISO systems. In section 4, we use two Monte Carlo simulations; the first is used to get a tuned value of the hyperparameter and the second one to compare the performance of the Minimax Regret for estimating the states of the linear dynamic system and two selected filtering. Finally, section V includes the conclusion and future works.

2. Minimax Regret for an uncertainty linear SISO system

Let us start by introducing the bounded discrete-time dynamic system (3)

$$x_{t+1} = a_t x_t,\ |x_t| \leq L,\ x_t \in \mathbb{R},\ t = 0, 1, 2, \dots \quad (3)$$

where x_t is the current (non-observable) state of the process, and each $a_t \in \mathbb{R}$ is the parameter of the real-valued system associated with the discrete-time instant $t \in \mathbb{N}$, and L is a parameter provided by a user. The Equation (3) is the unknown-but-bounded approach for the Equation (1a) in the traditional stochastic approach with an additive noise.

In parallel with the main dynamics (3), we next consider a discrete-time "observer" (space system) described in (4)

$$y_t = h_t x_t + \eta_t,\ t = 0, 1, 2, \dots \quad (4)$$

here, $h_t \in \mathbb{R}$ represents the known observer gain at time t, and $\eta_t \in \mathbb{R}$ denotes the disturbance term. Note that η_t characterizes the natural noise in the corresponding observation (4). With Equation (5), we assume that stochastic processes η_t are independent, centered and having known variance r_t for all $t, s = 0, 1, \dots$

$$E\{\eta_t\} = 0,\ E\{\eta_t \eta_s\} = \delta_{ts} r_t \quad (5)$$

where δ_{ts} is the delta Kronecker function.

An estimate of state x_t given by (3), which is linear concerning the last observation y_t, may be represented by (6)

$$\widetilde{x}_t = K_t y_t \tag{6}$$

where $K_t \in \mathbb{R}$. We will calculate the MSE of the estimator (6) by evaluating the bias $B(\widetilde{x}_t)$ and the variance $V(\widetilde{x}_t)$ of the estimator (6). By using (3) - (4) we have (7) and (8)

$$\begin{aligned} |B(\widetilde{x}_t)|^2 &= |x_t - E(\widetilde{x}_t)|^2 \\ &= a_{t-1}^2 x_{t-1}^2 (1 - K_t h_t)^2 \end{aligned} \tag{7}$$

$$V(\widetilde{x}_t) = E\left\{|\widetilde{x}_t - E\{\widetilde{x}_t\}|^2\right\} = K_t^2 r_t \tag{8}$$

then, the MSE is described in (9)

$$MSE(x_{t-1}, K_t) = (1 - K_t h_t)^2 a_{t-1}^2 x_{t-1}^2 + K_t^2 r_t. \tag{9}$$

Remark 1 (Non-observability). When the observation value h_t becomes zero, the MSE simplifies to $a_{t-1}^2 x_{t-1}^2 + K_t^2 r_t$. This simplification leads to an uninformative solution where K_t equals the zero, with no meaningful relationship between the unknown-but-bounded state x_{t-1} and the gain value K_t. Henceforth, we will assume that the system is observable for the remainder of our discussion.

We define, now, the regret function on (10)

$$\mathcal{R}(x_{t-1}, K_t) = MSE(x_{t-1}, K_t) - MSE^0 \tag{10}$$

where $MSE^0 = \min_{K_t} MSE\Big(x_{t-1}, K_t(x_{t-1})\Big)$. The regret (10) is the difference between the MSE using the estimator (6) and the smallest possible MSE attainable with an estimator $\widetilde{x}_t = K_t(x_{t-1})\, y_t$ when the state x_{t-1} is known.

The min-max regret filter is the value $\widetilde{x}_t$ of (6), where K_t is the solution of the min-max regret problem given by (11).

$$\min_{\widetilde{x}_t = K_t y_t} \max_{x_{t-1}^2 \leq L^2} \mathcal{R}(x_{t-1}, K_t). \tag{11}$$

To solve (11), we develop an explicit expression for MSE^0. First, we determine the estimator $\hat{x} = K_t(x_{t-1})\, y_t$ that minimizes the MSE (9) when x_{t-1} is known. To this end, we differentiate function (9) for K_t and equate to 0, which results in (12)

$$K_t(x_{t-1}) = \frac{h_t a_{t-1}^2 x_{t-1}^2 r_t^{-1}}{1 + h_t^2 a_{t-1}^2 x_{t-1}^2 r_t^{-1}} \tag{12}$$

substituting (12) into (9), the MSE^0 is given by (13)

$$MSE^0 = \frac{a_{t-1}^2 x_{t-1}^2}{1 + a_{t-1}^2 x_{t-1}^2 h_t^2 r_t^{-1}} \tag{13}$$

thus, substituting (13) into (10), we obtain (14), which is our final Regret function $\mathcal{R}(x_{t-1}, K_t)$.

$$a_{t-1}^2 x_{t-1}^2 (1 - K_t h_t)^2 + {K_t}^2 r_t - \frac{a_{t-1}^2 x_{t-1}^2}{1 + a_{t-1}^2 x_{t-1}^2 h_t^2 r_t^{-1}}. \tag{14}$$

3. Minimax Regret filter for uncertainty linear SISO model

The last section defined the concept of the Minimax Regret filter based on the gain value K_t and the Minimax Regret Problem (11). The main theorem for K_t is derived and proved.

Theorem 3.1 (Minimax Regret Gain Value)**.** *Let us denote the unknown-but-bounded linear real-valued state x_t in the system* (3) - (4)*, where $a_t \in \mathbb{R}$ and $h_t \in \mathbb{R}$ are known real-valued associated with the discrete-time system, $\eta_t \in \mathbb{R}$ denotes system uncertainties of the observer equation for each $t \in \mathbb{N}$. Then, K_t^* is the solution to the min-max* (15)

$$\min_{K_t} \max_{x_{t-1}^2 \leq L^2} \mathcal{R}(x_{t-1}, K_t) \tag{15}$$

where the regret function is defined in (14)*, it has the form described in* (16)

$$K_t^* = \frac{a_{t-1}^2 h_t}{\frac{r_t}{L^2} + a_{t-1}^2 h_t^2} \tag{16}$$

Proof. To get the solution to (15), first, we transform the minimax problem into a min problem. As the part $K_t^2 r_t$ of (14) is independent of x_{t-1}, we have the transformed problem (17)

$$\min_{K_t} \Bigg[K_t^2 r_t + \max_{x_{t-1}^2 \leq L^2} \left(x_{t-1}^2 a_{t-1}^2 (1 - K_t h_t)^2 - \frac{x_{t-1}^2 a_{t-1}^2}{1 + x_{t-1}^2 a_{t-1}^2 h_t^2 r_t^{-1}}\right)\Bigg] \tag{17}$$

Let us define the sub-problem of maximizing on $x_{t-1}^2 \leq L^2$ the function $f(x_{t-1}, K_t)$ defined in (18)

$$x_{t-1}^2 a_{t-1}^2 (1 - K_t h_t)^2 - \frac{x_{t-1}^2 a_{t-1}^2}{1 + x_{t-1}^2 a_{t-1}^2 h_t^2 r_t^{-1}} \tag{18}$$

As the function given in (18) is a continuous function defined on a compact set $\left\{x_{t-1} : x_{t-1}^2 \leq L^2\right\}$, by Weierstrass theorem, it attains its maximum on the closed interval $[-L, L]$ for any K_t fixed.

The Equation (19) is the Lagrange function for the sub-problem with function (18)

$$\Lambda(x_{t-1}, \lambda) = (c_{1t} - \lambda)\, x_{t-1}^2 - \frac{x_{t-1}^2 a_{t-1}^2}{1 + x_{t-1}^2 c_{2t}} + \lambda L^2 \tag{19}$$

where $c_{1t} = a_{t-1}^2 (1 - K_t h_t)^2$, and $c_{2t} = a_{t-1}^2 h_t^2 r_t^{-1}$. The supremum of (19) is attainable iff $c_{1t} - \lambda \leq 0$, and $q(\lambda) =$

$\sup_{x_{t-1}\in\mathbb{R}} \Lambda(x_{t-1}, K_t, \lambda) = \lambda L^2$ is the dual function. The Lagrange dual problem is described in (20)

$$\begin{aligned} \min_{\lambda} \quad & \lambda L^2 \\ \text{s.t.} \quad & -\lambda \leq 0 \\ & a_{t-1}^2 (1 - K_t h_t)^2 - \lambda \leq 0. \end{aligned} \tag{20}$$

Using (20), the Minimax Regret optimization problem (17) becomes

$$\min_{K_t} \left[K_t^2 r_t + \min \begin{array}{cc} & \lambda L^2 \\ \text{s.t.} & a_{t-1}^2 (1 - K_t h_t)^2 - \lambda \leq 0 \\ & \lambda \geq 0 \end{array} \right]$$

and it is equivalent to (21)

$$\begin{aligned} \min \quad & K_t^2 r_t + \lambda L^2 \\ \text{s.t.} \quad & a_{t-1}^2 (1 - K_t h_t)^2 - \lambda \leq 0 \\ & \lambda \geq 0, K_t \text{ free.} \end{aligned} \tag{21}$$

We next define the Karush-Kuhn-Tucker (KKT) conditions for (21) when we introduce the auxiliary decision variables $K_t = K_t^+ - K_t^-$; $K_t^+, K_t^- \geq 0$ and the Lagrange function

$$\mathcal{L}\left(K_t^+, K_t^-, \lambda\right) = K_t^2 r_t + \lambda L^2 + \sum_{i=1}^{4} \mu_i g_i(K_t^+, K_t^-, \lambda)$$

where $g_1\left(K_t^+, K_t^-, \lambda\right) = a_{t-1}^2 \left[1 - \left(K_t^+ - K_t^-\right) h_t\right]^2 - \lambda$, $g_2\left(K_t^+, K_t^-, \lambda\right) = -K_t^+$, $g_3\left(K_t^+, K_t^-, \lambda\right) = -K_t^-$, and $g_4\left(K_t^+, K_t^-, \lambda\right) = -\lambda$. Then, the Lagrangian optimality

$$\nabla F\left(K_t^+, K_t^-, \lambda\right) + \sum_{i=1}^{4} \mu_i \nabla g_i\left(K_t^+, K_t^-, \lambda\right) = 0$$

is equivalent to (22a) - (22c)

$$2K_t r_t - 2\mu_1 a_{t-1}^2 \left[1 - K_t h_t\right] h_t - \mu_2 = 0 \tag{22a}$$

$$-2K_t r_t + 2\mu_1 a_{t-1}^2 \left[1 - K_t h_t\right] h_t - \mu_3 = 0 \tag{22b}$$

$$L^2 - \mu_1 - \mu_4 = 0. \tag{22c}$$

The slackness equations $\mu_i g_i\left(K_t^+, K_t^-, \lambda\right) = 0$, $i = 1, 2, 3, 4$ are given by (23a)-(23d)

$$\mu_1 \left[a_{t-1}^2 \left[1 - \left(K_t^+ - K_t^-\right) h_t\right]^2 - \lambda\right] = 0 \tag{23a}$$

$$\mu_2 K_t^+ = 0 \tag{23b}$$

$$\mu_3 K_t^- = 0 \tag{23c}$$

$$\mu_4 \lambda = 0 \tag{23d}$$

Adding (22a) and (22b), and joint to dual feasibility condition $\mu_2, \mu_3 \geq 0$, we get (24)

$$\mu_2 = \mu_3 = 0 \tag{24}$$

thus (23b) and (23c) always hold.

Inserting (24) into (22a) and (22b) simplifies to (25)

$$\begin{aligned} 2\left(K_t^+ - K_t^-\right) r_t - 2\mu_1 a_{t-1}^2 \left[1 - \left(K_t^+ - K_t^-\right) h_t\right] h_t = 0 \\ -2\left(K_t^+ - K_t^-\right) r_t + 2\mu_1 a_{t-1}^2 \left[1 - \left(K_t^+ - K_t^-\right) h_t\right] h_t = 0 \end{aligned} \tag{25}$$

Case 1. Assuming $\mu_1 > 0$, and the condition (23a), we get the key condition (26)

$$a_{t-1}^2 \left[1 - \left(K_t^+ - K_t^-\right) h_t\right]^2 = \lambda. \tag{26}$$

Now, we consider the following subcases.

Case (a) $\mu_4 > 0$, then, by (23d) we have $\lambda = 0$. Also, (23a) implies that $1 - \left(K_t^+ - K_t^-\right) h_t = 0$ by (26), and $\left(K_t^+ - K_t^-\right) = \frac{1}{h_t} \neq 0$ and (25) does not hold. So, the only possible case is $\mu_4 = 0$. $\lambda = 0$, similarly to the latter argument is not possible.
Case (b) $\lambda > 0$. Using the result of Case (a), by (22c), we get $\mu_1 = L^2$, and substituting μ_1 into (25) and solving it, we get (27)

$$\left(K_t^+ - K_t^-\right)^* = K_t^* = \frac{L^2 a_{t-1}^2 h_t}{r_t + L^2 a_{t-1}^2 h_t^2} \tag{27}$$

and by (26) we obtain (28)

$$\lambda^* = a_{t-1}^2 \left[1 - \frac{L^2 a_{t-1}^2 h_t^2}{r_t + L^2 a_{t-1}^2 h_t^2}\right]^2 > 0. \tag{28}$$

Case 2. The only remaining case is $\mu_1 = 0$. This assumption and (22c) imply $\mu_4 = L^2 > 0$. By Case (a), it is not possible.

Thus the only solution of (15) is given by the Equations (27), and (28). □

Theorem 3.1 reduces the problem of minimizing the worst-case regret in the framework of univariate dynamical linear systems to a closed form given by (16). The univariate Minimax Regret estimator depends on the bounding parameter L. From (16) the larger variance r_t, the closer to zero is the real-valued of the gain, which gives a character of robustness to the Minimax Regret filter in the presence of outliers for each constant L. (16) shows interesting additional cases, one is when $r_t \to 0$, the real-valued gain K_t goes to the naive solution $\frac{1}{h_t}$, as well as $r_t \neq 0$ and $L \to \infty$. Finally, when $L \to 0^+$, K_t goes to zero.

4. Simulation study

We now present a simulation study that illustrates the performance of the Minimax Regret estimator. We use two metrics proposed in [29] to assess the performance of our proposal. We consider the problem of estimating a one-dimensional state from a dynamical linear system, which is obtained by simulating the system (3)-(4), whose dynamic state space equations are given by

$$x_{t+1} = -0.98x_t,\ |x_t| \le L,\ x_0 \in \mathbb{R}$$
$$y_t = 2x_t + \eta_t,\ t \in \mathbb{N}$$

First of all, we need to tune the parameter L. We use a Monte Carlo simulation to select this parameter. This is described in subsection 4.1. Second, we compare the performance of the Minimax Regret with other methodologies, namely, the classic Kalman Filter (KF) and the Innovative and/or Additive Outlier Robust Kalman Filtering is implemented in the RobKF package (RKF) in the R software [30]. We use another Monte Carlo simulation to assess the performance of the following methodologies: the Minimax Regret filter with the L value selected in subsection 4.1, the Classic KF, and RKF, which is described in subsection 4.2.

We perform one simulation of two sequences of $N = 100$ points with starting state point $x_0 = -15$. The measurement noise without external outliers is assumed to follow three different distributions; those are, Gaussian distribution with zero mean and variance two, Student's t-distribution with three degrees of freedom, and Laplace distribution using a location parameter of zero and scale parameter of two. The total Monte Carlo trials are $M = 50$. Finally, we introduce a normal error with zero mean and variance one in the state x_t to simulate the fact that two different states $\overline{x}_t$ and $x_t^\star$ can produce two close space data $\overline{y}_t$ and $y_t^\star$ and the non-observable state which is unknown-but-bounded.
Let $x_t^{d,m}$ and $\widetilde{x}_t^{d,m}(L)$ be the true state data and the L-minimax filtered data for the distribution d in the set of the three selected statistical distribution D in the m-th Monte Carlo trial for $t = 1, \ldots, 100$, respectively. We use the generalized metrics (29) and (30), similar as [29]

$$MSE_d^L(m) = \frac{1}{N}\sum_{t=1}^{N}\left(x_t^{d,m} - \widetilde{x}_t^{d,m}(L)\right)^2 \qquad (29)$$

for each $m = 1, \ldots, M$, and

$$RMSE_d^L(t) = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(x_t^{d,m} - \widetilde{x}_t^{d,m}(L)\right)^2} \qquad (30)$$

To ensure the reliability of data analysis, robustness measures play a crucial role in guiding the development of methods that can adapt effectively to variations in the data. This is essential for preventing undesired responses, such as erratic behavior, in the presence of outliers. In addition to conventional loss functions, we propose the incorporation of mean Huber error (MHE) and root mean Huber error ($RMHE$) to further enhance the robustness of data analysis methods. This function is a smooth transition between quadratic and linear penalties that helps mitigate the influence of outliers, leading to more accurate and reliable results when dealing with data containing outliers [31–33], which are supported in the Huber loss function given below

$$\mathcal{L}_\delta(e) = \begin{cases} \frac{1}{2}e^2 & \text{if } |e| \le \delta \\ \delta(|e| - \frac{1}{2}\delta) & \text{if } |e| > \delta \end{cases}$$

where $e = x_t^{d,m} - \widetilde{x}_t^{d,m}(L)$, and δ is a user-defined threshold between zero and ∞. Therefore, the MHE and $RMHE$ are defined as

$$MHE_{d,\delta}^L(m) = \frac{1}{N}\sum_{t=1}^{N}\mathcal{L}_\delta\left(x_t^{d,m} - \widetilde{x}_t^{d,m}(L)\right)^2 \qquad (31)$$

$$RMHE_{d,\delta}^L(t) = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\mathcal{L}_\delta\left(x_t^{d,m} - \widetilde{x}_t^{d,m}(L)\right)^2} \qquad (32)$$

for each $t = 1, \ldots, N$, and each of three selected distributions indexed by d. Those are the conditions for both simulations in the following subsection 4.1 and 4.2.

4.1 L tune simulation

To ensure the Minimax Regret filter performs optimally, its parameter L needs to be selected. This study identifies the optimal L^* by evaluating the reliability of state estimates obtained under various L values. Both MSE and MHE serve as metrics for estimating reliability. The optimal L^* is the value that minimizes both MSE and MHE loss functions. This is achieved by reformulating Equations (29) and (31) as functions of L, resulting in (33) and (34), respectively. These reformulated equations treat L as the decision variable, enabling the efficient identification of the L^* that yields the most reliable state estimates based on both MSE and MHE criteria.

$$MSE_{d,m}(L) = \frac{1}{N}\sum_{t=1}^{N}\left(x_t^{d,m} - \widetilde{x}_t^{d,m}(L)\right)^2 \qquad (33)$$

$$MHE_{d,m}^\delta(L) = \frac{1}{N}\sum_{t=1}^{N}\mathcal{L}_\delta\left(x_t^{d,m} - \widetilde{x}_t^{d,m}(L)\right)^2. \qquad (34)$$

To achieve this, the study adopts a Monte Carlo approach with extensive simulations. The Monte Carlo data were generated using a seed set to 123. A descriptive analysis is conducted to extract insights into the randomness exhibited by the functions $MSE_{d,m}(L)$ and $MHE_{d,m}^\delta(L)$ with $\delta = 1.345$, in the definition (33) and (34), respectively. As a criterion to select the L parameter, the $mean$ and

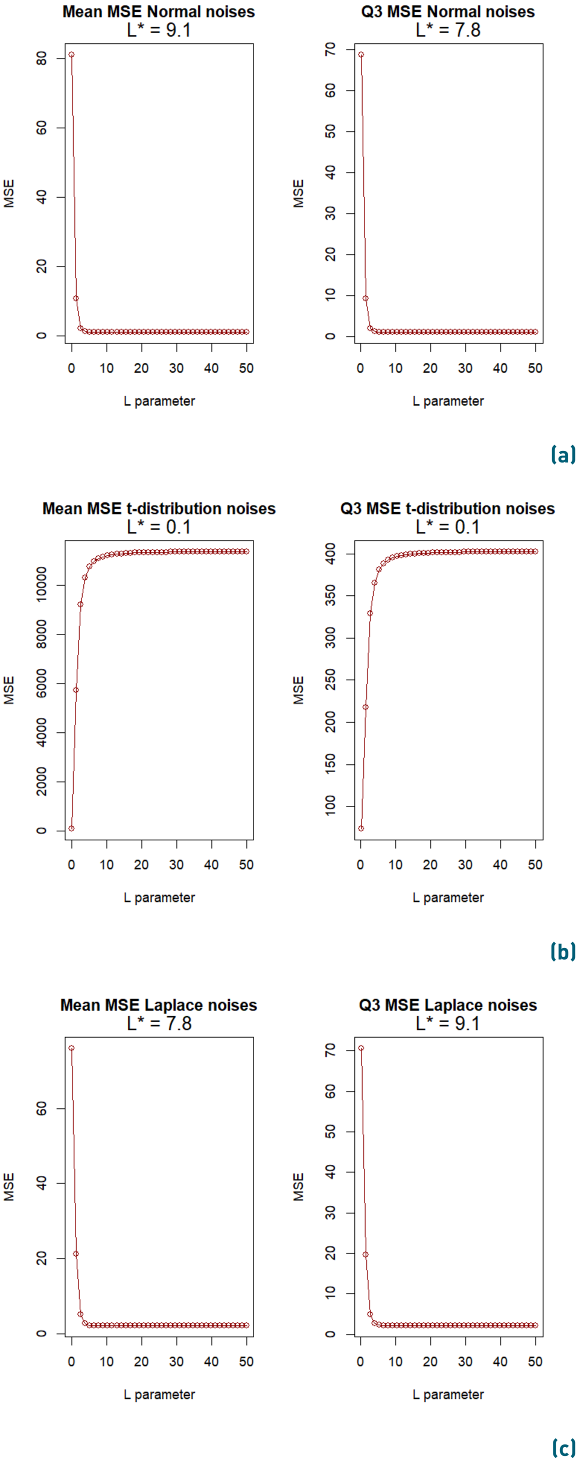

Figure 1 Hyperparameter tune under MSE loss function

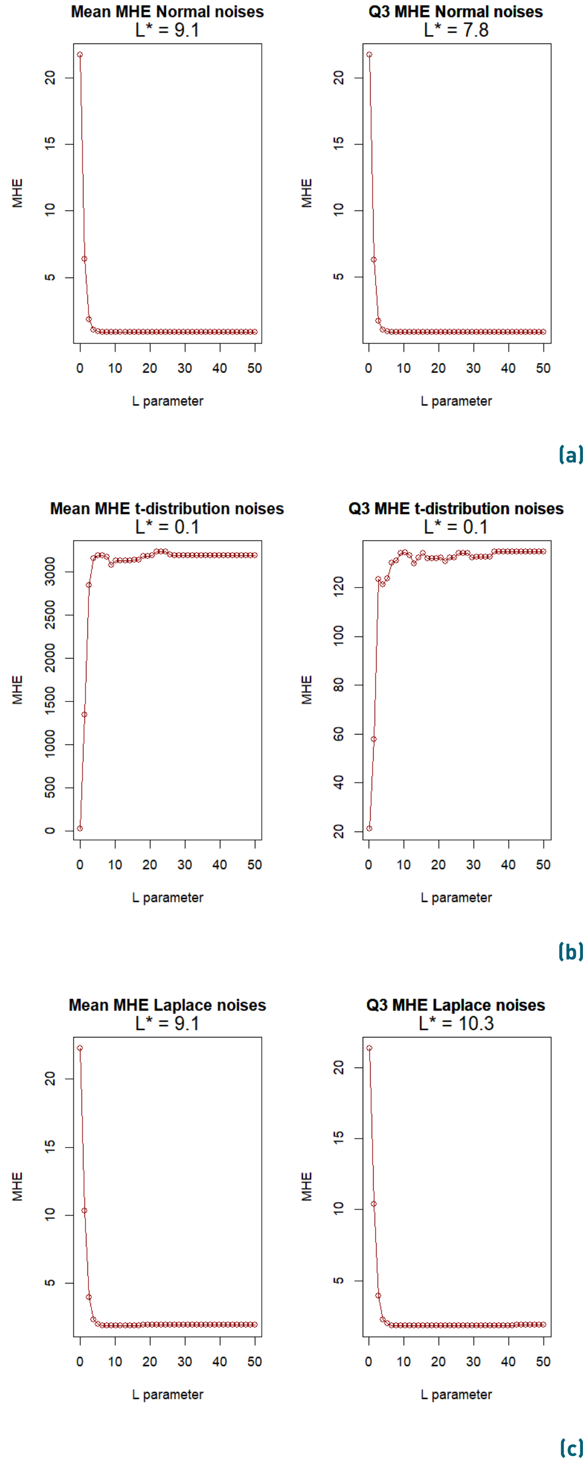

Figure 2 Hyperparameter tune under MHE loss function

third quartile ($Q3$) of the $MSE_{d,m}(L)$ in the Equation (33) and for $MHE^{\delta}_{d,m}(L)$ in Equation (34), overall M simulations were calculated for each distribution $d \in \mathrm{D}$ to highlight a range of e□icient values of L.

The Grid Search method, a common hyperparameter tuning technique in machine learning, was employed to optimize the L value. This approach is particularly effective for models with a limited number of hyperparameters, as it exhaustively evaluates all possible parameter combinations within a pre-defined grid. This method is well-suited for scenarios where the number of hyperparameters is manageable, allowing for a comprehensive exploration of the parameter space to identify the optimal configuration [34]. The Grid Search was performed on parameter L with 100 equally spaced points between 0 and 50. For each value of L, the Monte Carlo simulations were conducted across statistical distributions. The $mean$ and $Q3$ of both $MSE_{d,m}(L)$ and the $MHL_{d,m}(L)$ were calculated for each distribution $d \in \mathrm{D}$ and presented in Figures. 1 and 2, respectively. These figures show the values that minimize the $MSE_{d,m}(L)$ and $MHE^{1.345}_{d,m}(L)$ for each distribution and statistics. Those values mean that the state estimates gain higher accuracy when the Minimax Regret filter is evaluated at those L^{*} values. It is possible to see that both loss criteria decrease exponentially for the normal and Laplace distributions, while the t-distribution exhibits a concave form, getting the minimum value for both metrics at $L^{*} = 0.1$.

4.2 Minimax Regret filter performance

After the optimal value L^{*} had been obtained in the set of distributions D in the immediately preceding subsection 4.1, we generated a new data set with the same former conditions but the seed set to 53.

The performance comparison of state estimates by the filters is based on two classic metrics: MSE for each Monte Carlo run and RMSE for each t, represented by Equations (29) and (30), respectively. Additionally, robust metrics given by Equations (31) and (32) for MHE and RMHE, respectively, were employed.

The statistical analysis, depicting the evolution of MSE and RMSE for each distribution and statistic, is presented in Figure 3 and Figure 4. For normal noises, the KF and the Minimax Regret filter exhibit marginal differences in MSE estimation, while the RKF performs poorly. However, in terms of RMSE, the Minimax Regret filter outperforms the other two filters significantly. Regarding the MHE metric for normal noises, both statistics of the Minimax Regret filter show superior performance compared to the other filters. Although exhibiting higher variation than MHE,

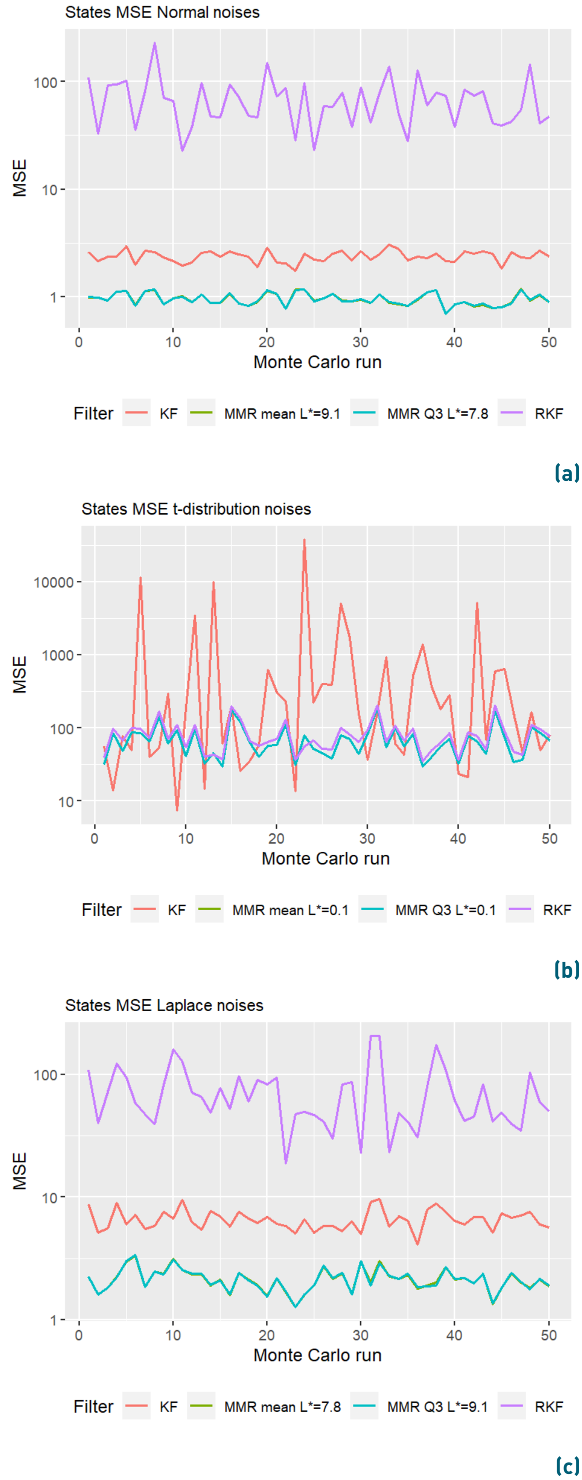

Figure 3 MSE Monte Carlo runs by distribution

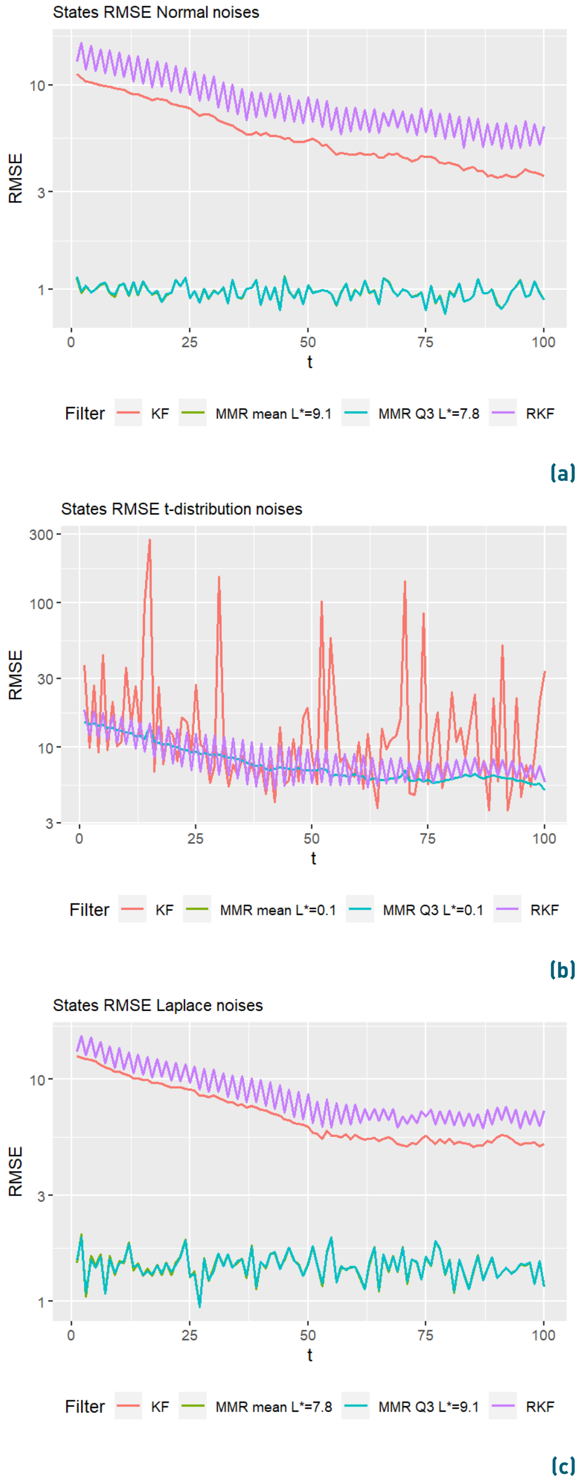

Figure 4 RMSE Monte Carlo runs by distribution

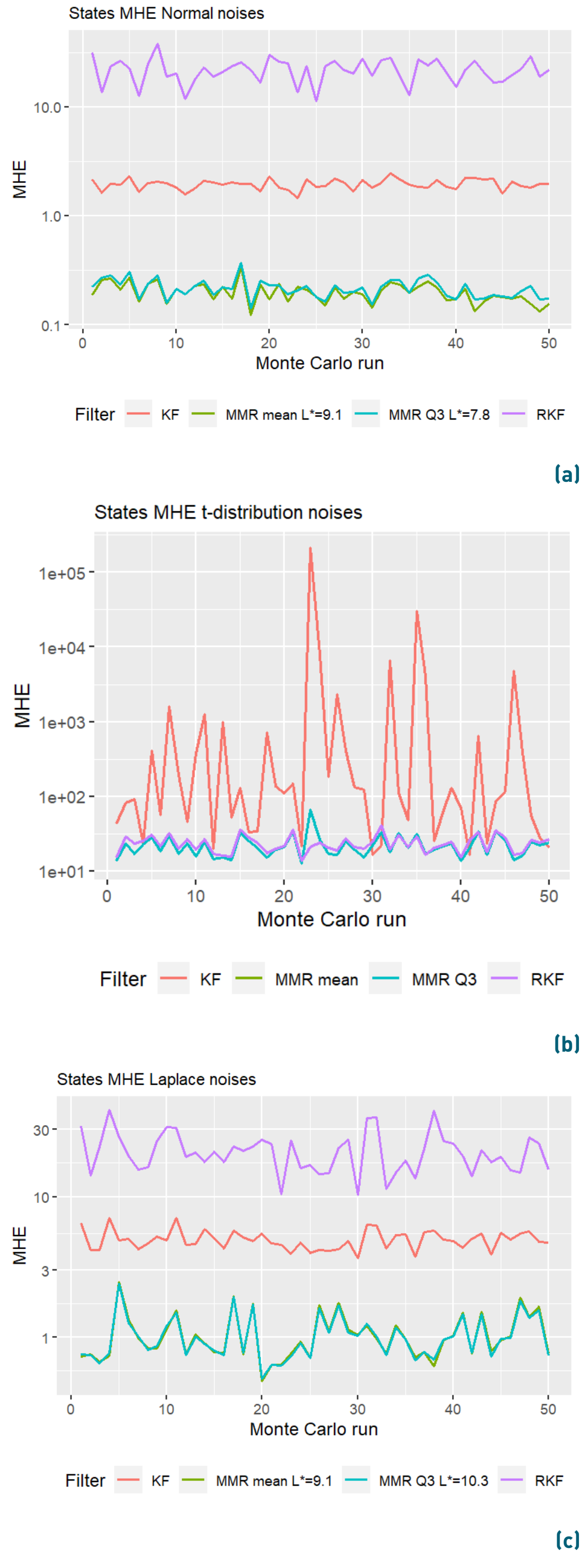

Figure 5 MHE Monte Carlo runs by distribution

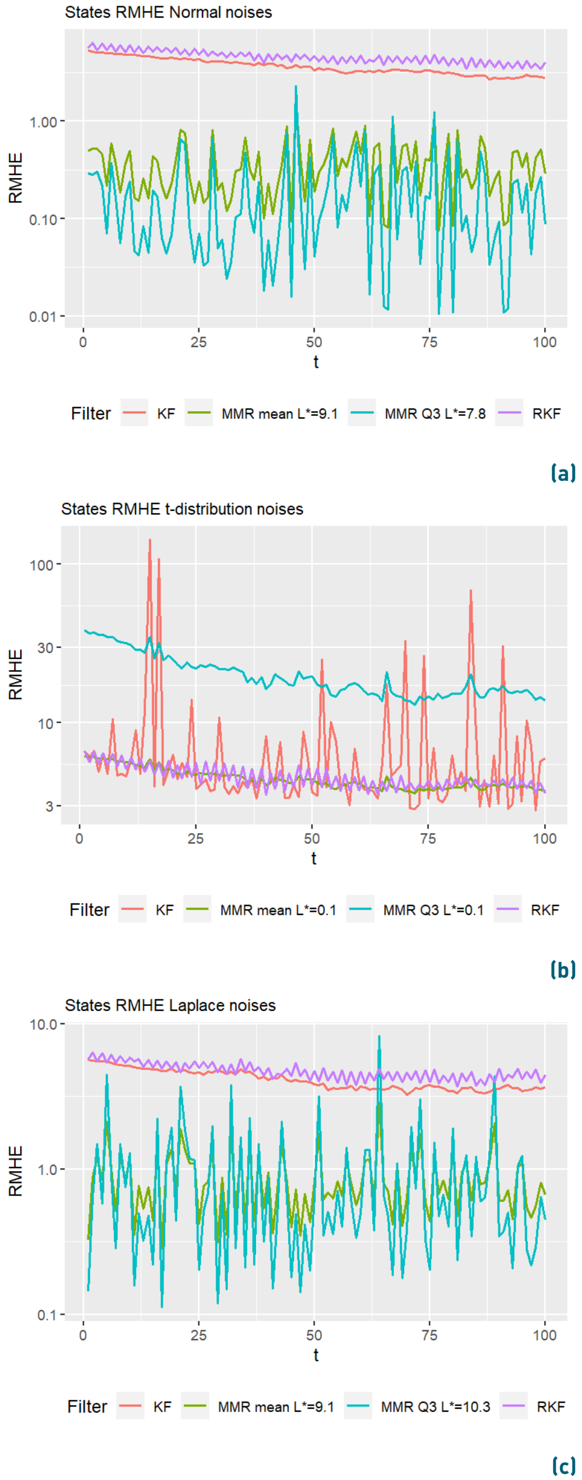

Figure 6 RMHE Monte Carlo runs by distribution

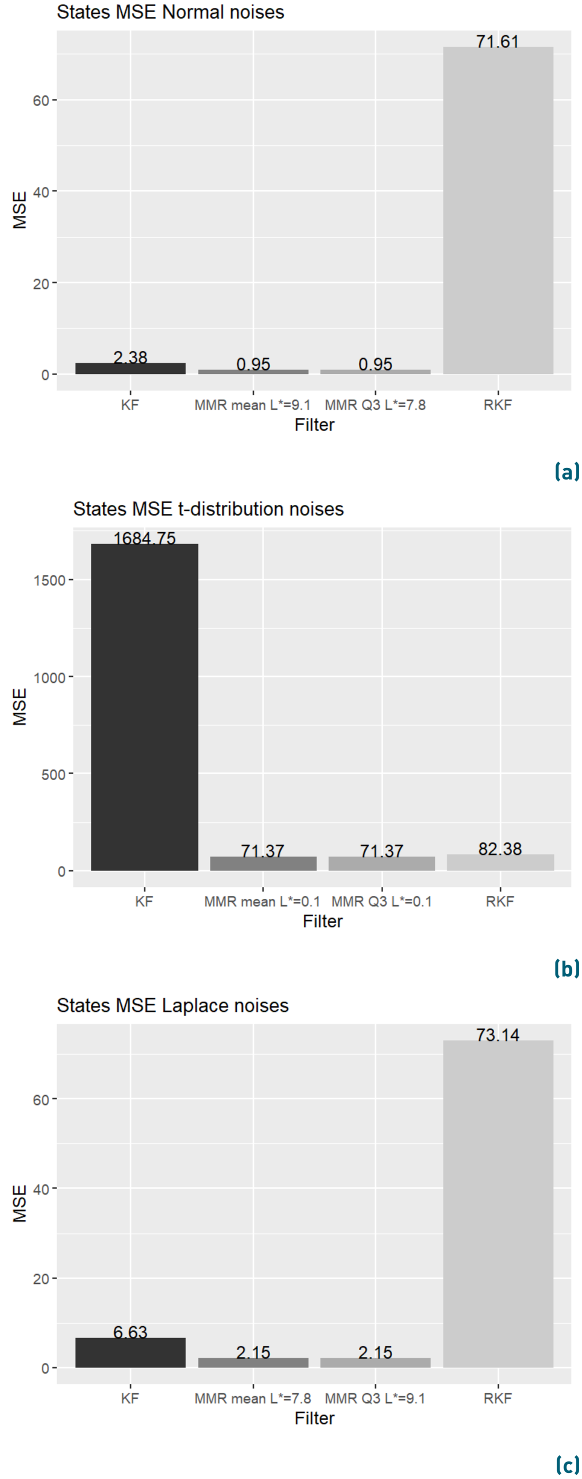

Figure 7 MSE histogram by distribution

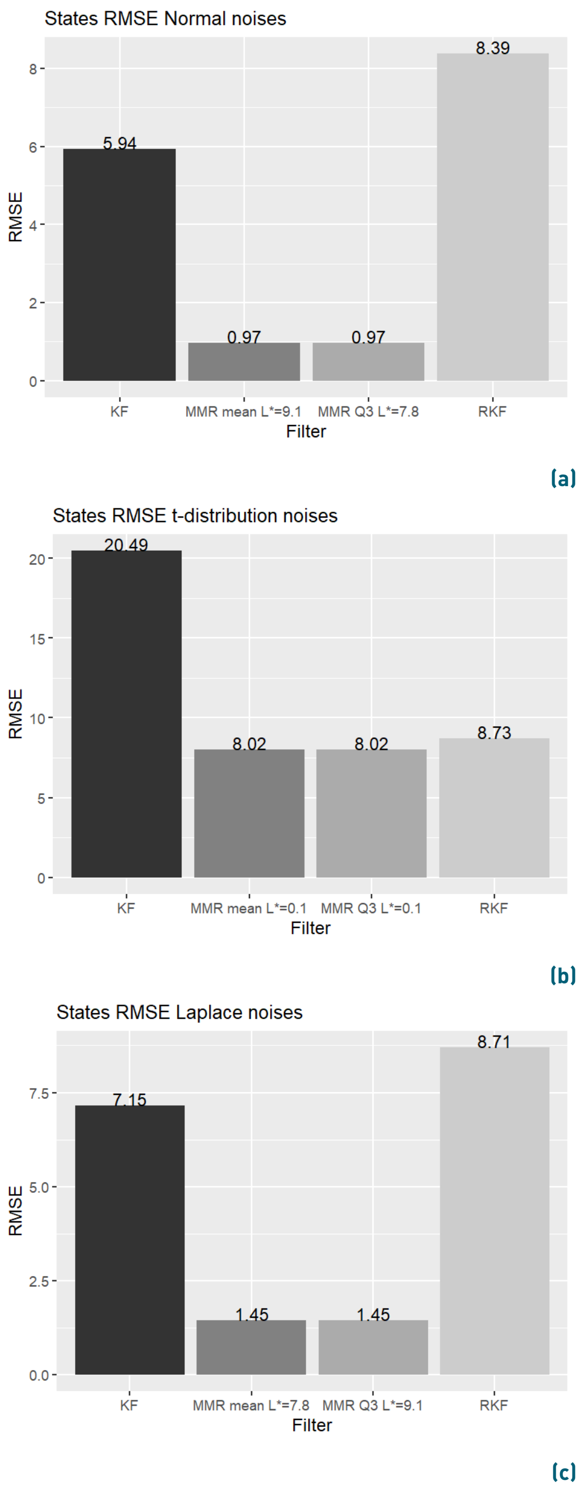

Figure 8 RMSE histogram by distribution

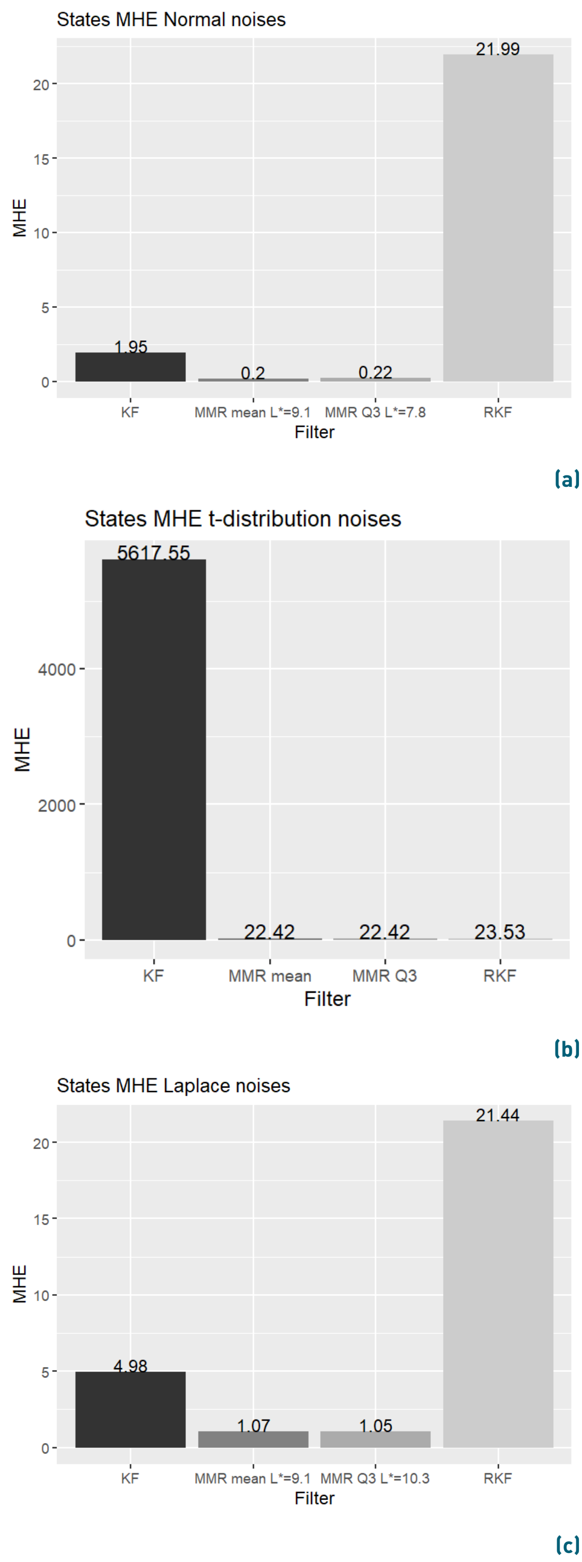

Figure 9 MHE histogram by distribution

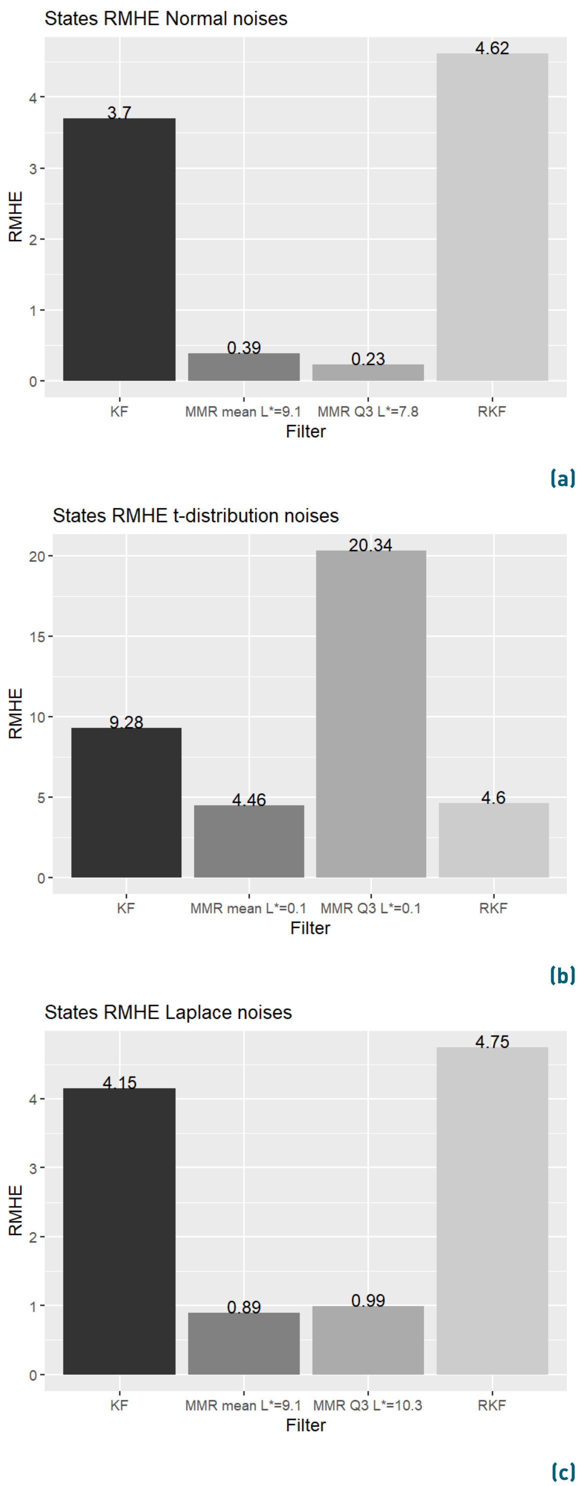

Figure 10 RMHE histogram by distribution

RMHE consistently outperforms the KF and RKF.

For processes characterized by heavy-tailed distributions, such as the t-distribution, the RKF outperforms the classical KF in terms of MSE. However, while the RKF's estimates are comparable to those of the Minimax Regret filter in MSE, the latter demonstrates superior performance in RMSE. Notably, the Minimax Regret filter offers an additional advantage with a closed-form solution for one-dimensional linear systems. This result is illustrated more clearly in Figures 7b and 8b. In this distribution, the loss values in Figure 5b and Figure 9b are very close, leading to a similar conclusion for MHE. Nevertheless, Figure 6b and Figure 10b indicate that the Minimax Regret under the $Q3$ statistic is slightly less competitive than the RKF.

For the Laplace distribution, as shown in Figures 6c to 10c, the Minimax Regret filter consistently outperforms both the KF and RKF across various metrics, including MSE, MRSE, MHE, and RMHE, evaluated for both $mean$ and $Q3$ statistics. Notably, the Minimax Regret filter exhibits a remarkable performance advantage, achieving up to 97% lower MSE and up to 83% lower RMSE compared to the RKF. These results underscore the superior estimation accuracy of the Minimax Regret filter, particularly in the presence of heavy-tailed noise. This finding has significant implications for applications where robust state estimation is crucial.

5. Conclusions and future works

We addressed the problem of estimating an unknown-but-bounded real-valued state and a stochastic real-valued space in the uncertainty linear SISO model (3) - (4) in the minimax regret framework. Each state in this model belongs to the uncertainty set $\{x_{t-1} : x_{t-1}^2 \leq L^2\}$, representing a novel approach that diverges from existing literature, considering prior knowledge of the probability distribution or the uncertainty described by a group of weighted probability measures controlling the state.
We developed a new estimator of the states for a one-dimensional dynamical linear model based on minimizing the worst-case regret, which is defined as the difference between the MSE of the estimator and the best possible MSE attainable with a dynamical linear estimator that knows the state value x_{t-1} based on a linear operator. We opt for a linear operator instead of an affine operator, specifically excluding the use of an affine function like $x_{t-1} = K_t y_t + b_t$. This choice is deliberate, motivated by our interest in scenarios where the bias deviates from zero. Using optimization tools, such as Lagrange theory, duality, and KKT conditions, our main contribution is the theorem on the closed form of the gain value K_t.

It was shown, with a simulation study, how this novel concept is applied. Furthermore, the simulation study in section 4.2 provides evidence of the Minimax Regret performance by comparing it with the KF and the RKF. The simulation study showed that our methodology is competitive or better in the selected metric for the studied distributions. Interesting directions for future research are to study how to define mathematical bounds for the L hyperparameter independent of the simulation distribution, how to estimate the covariance matrix of the stochastic part of the model by the Minimax Regret approach, how to extend the Minimax Regret filter to the multidimensional case, and define a methodology that allows the estimation of parameters a_t, h_t, and r_t using the Minimax Regret concept.

Declaration of competing interest

We declare that we have no significant competing interests, including financial or non-financial, professional, or personal interests interfering with the full and objective presentation of the work described in this manuscript.

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Author contributions

J. Perea-Arango: Methodology, software, validation, formal analysis and investigation. P. Graczyk: Conceptualization and formal analysis. J. P. Fernández-Gutiérrez: Conceptualization, methodology and formal analysis

Data availability statement

The authors confirm that the data used in this study were simulated, and that the findings can be reproduced by following the assumptions and procedures outlined in the paper.

References

[1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering*, vol. 82, no. 1, 1960. [Online]. Available: https://doi.org/10.1115/1.3662552

[2] J. Iqbal, M. Rehman-Saad, A. Mahmood-Tahir, and A. Malik, "State estimation technique for a planetary robotic rover," *Revista Facultad De Ingenieria Universidad De Antioquia*, no. 73, Oct-Dec 2014. [Online]. Available: https://doi.org/10.17533/udea.redin.17275

[3] Q. Ge, J. Ma, H. He, H. Li, and G. Zhang, "A basic smart linear kalman filter with online performance evaluation based on observable degree," *Applied Mathematics and Computation*, vol. 367, 2020. [Online]. Available: https://doi.org/10.1016/j.amc.2019.124603

[4] Y. Li, X. Song, Z. Zhang, D. Zhao, and Z. Wang, "h^∞ deconvolution filter design for uncertain linear discrete time-variant systems: A krein space approach," *Applied Mathematics and Computation*, vol. 361, 2019. [Online]. Available: https://doi.org/10.1016/j.amc.2019.05.015

[5] T. S. Badings, A. Abate, N. Jansen, D. Parker, H. A. Poonawala, and M. Stoelinga, "Sampling-based robust control of autonomous systems with non-gaussian noise," *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 36, no. 9, Jun. 2022. [Online]. Available: https://doi.org/10.1609/aaai.v36i9.21201

[6] M. Mintz, "A note on minimax estimation and kalman filtering," *IEEE Transactions on Automatic Control*, vol. 23, no. 4, 1978. [Online]. Available: https://doi.org/10.1109/TAC.1969.1099239

[7] M. Mintz, "A kalman filter as a minimax estimator," *Journal of Optimization Theory and Applications*, vol. 9, no. 1, 1972. [Online]. Available: https://doi.org/10.1007/BF00932347

[8] N. Castaño, J. P. Fernández-Gutiérrez, V. Azhmyakov, and P. Graczyk, "Non-parametric identification of upper bound covariance matrices for min-sup robust kalman filter: Application to the ar case," in *IFAC-PapersOnLine*, vol. 55, no. 40, 2022. [Online]. Available: https://doi.org/10.1016/j.ifacol.2023.01.068

[9] M. J. Morris, "The kalman filter: A robust estimator for some classes of linear quadratic problems," *IEEE Transactions on Information Theory*, vol. 22, no. 5, 1976. [Online]. Available: https://doi.org/10.1109/TIT.1976.1055611

[10] Y. Sholeh and K. Pelckmans, "Worst-case prediction performance analysis of the kalman filter," *IEEE Transactions on Automatic Control*, vol. 63, no. 6, 2018. [Online]. Available: https://doi.org/10.1109/TAC.2017.2757908

[11] A. Pankov and G. Miller, "Minimax filter for statistically indeterminate stochastic differential system," *IFAC Proceedings Volumes*, vol. 38, no. 1, 2005. [Online]. Available: https://doi.org/10.3182/20050703-6-CZ-1902.00389

[12] E. Gyurkovics and T. Takács, "Robust energy-to-peak filter design for a class of unstable polytopic systems with a macroeconomic application," *Applied Mathematics and Computation*, vol. 420, 2022. [Online]. Available: https://doi.org/10.1016/j.amc.2021.126729

[13] Y. C. Eldar, A. Ben-Tal, and A. Nemirovski, "Minimax regret estimation in linear models," in *2004 IEEE International Conference on Acoustics, Speech, and Signal Processing*, vol. 2, 2004. [Online]. Available: https://doi.org/10.1109/ICASSP.2004.1326219

[14] B. Hassibi, A. H. Sayed, and T. Kailath, "Linear estimation in krein spaces-theory," *IEEE Transactions on Automatic Control*, vol. 41, no. 1, pp. 18–33, 1996. [Online]. Available: https://doi.org/10.1109/9.481605

[15] B. Hassibi and T. Kailath, "H/sup /spl infin// bounds for least-squares estimators," *IEEE Transactions on Automatic Control*, vol. 46, no. 2, 2001. [Online]. Available: https://doi.org/10.1109/9.905700

[16] M. M. J. Mirza, "A modified kalman filter for non-gaussian measurement noise," in *Communication Systems and Information Technology*, M. Ma, Ed. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011. [Online]. Available: https://doi.org/10.1007/978-3-642-21762-3_52

[17] X. Sun, J. Duan, X. Li, and X. Wang, "State estimation under non-gaussian levy noise: A modified kalman filtering method," *arXiv*

e-prints, 2013. [Online]. Available: https://ui.adsabs.harvard.edu/abs/2013arXiv1303.2395S

[18] V. Azhmyakov, "Optimal control of a well-stirred bioreactor in the presence of stochastic perturbations," *Informatica*, vol. 13, no. 2, 2002. [Online]. Available: https://doi.org/10.3233/INF-2002-13201

[19] J. C. Spall, "The kantorovich inequality for error analysis of the kalman filter with unknown noise distributions," *Technical Communique*, vol. 31, no. 10, 1995. [Online]. Available: https://doi.org/10.1016/0005-1098(95)00069-9

[20] J. L. Maryak, J. C. Spall, and B. D. Heydon, "Use of the kalman filter for inference in state-space models with unknown noise distributions," *IEEE Transactions on Automatic Control*, vol. 46, no. 2, 2001. [Online]. Available: https://doi.org/10.1109/TAC.2003.821415

[21] H.-S. Ahn, Y.-S. Kim, and Y. Chen, "An interval kalman filtering with minimal conservatism," *Applied Mathematics and Computation*, vol. 218, no. 18, 2012. [Online]. Available: https://doi.org/10.1016/j.amc.2012.02.050

[22] Y. Liu, Y. Zhao, and F. Wu, "Ellipsoidal state-bounding-based set-membership estimation for linear system with unknown-but-bounded disturbances," *IET Control Theory & Applications*, vol. 10, no. 4, 2016. [Online]. Available: https://doi.org/10.1049/iet-cta.2015.0654

[23] Z. Wang, G. Xu, Z. Liu, Y. Wang, and Z. Ji, "Orthotopic-filtering-based fault diagnosis algorithms for nonlinear systems with slowly varying faults," *Journal of the Franklin Institute*, vol. 357, 2020. [Online]. Available: https://doi.org/10.1016/j.jfranklin.2020.03.033

[24] L. Savage, "The theory of statistical decision," *Journal of the American Statistical Association*, vol. 46, 1951. [Online]. Available: https://doi.org/10.1080/01621459.1951.10500768

[25] N. Oliver, F. Perez-Cruz, S. Kramer, J. Read, and J. Lozano, *Machine Learning and Knowledge Discovery in Databases. Research Track European Conference, ECML PKDD 2021, Bilbao, Spain, September 13–17, 2021, Proceedings, Part III: European Conference, ECML PKDD 2021, Bilbao, Spain, September 13–17, 2021, Proceedings, Part III*, 2021. [Online]. Available: https://doi.org/10.1007/978-3-030-86486-6

[26] A. No and T. Weissman, "Minimax filtering regret via relations between information and estimation," *IEEE Transactions on Information Theory*, vol. 60, no. 8, 2014. [Online]. Available: https://doi.org/10.1109/TIT.2014.2323069

[27] J. Y. Halpern and S. Leung, "Minimizing regret in dynamic decision problems," *Theory and Decision*, vol. 81, 2016. [Online]. Available: https://doi.org/10.1007/s11238-015-9526-8

[28] Y. Eldar, A. Ben-Tal, and A. Nemirovski, "Linear minimax regret estimation of deterministic parameters with bounded data uncertainties," *IEEE Transactions on Signal Processing*, vol. 52, no. 8, 2004. [Online]. Available: https://doi.org/10.1109/TSP.2004.831144

[29] L. Chang, B. Hu, G. Chang, and A. Li, "Robust derivative-free kalman filter based on huber's m-estimation methodology," *Journal of Process Control*, vol. 23, no. 10, 2013. [Online]. Available: https://doi.org/10.1016/j.jprocont.2013.05.004

[30] A. T. M. Fisch, D. Grose, I. A. Eckley, P. Fearnhead, and L. Bardwell, *RobKF: Innovative and/or Additive Outlier Robust Kalman Filtering*, 2022, r package version 1.0.2. [Online]. Available: https://doi.org/10.48550/arXiv.2007.03238

[31] J. Fan, R. Li, C.-H. Zhang, and H. Zou, "Statistical foundations of data science," in *Statistical Foundations of Data Science*. Chapman and Hall/CRC, 2021.

[32] D. Akkaya and M. C. Pınar, "Minimizers of sparsity regularized huber loss function," *Journal of Optimization Theory and Applications*, vol. 187, 2020. [Online]. Available: https://doi.org/10.1007/s10957-020-01745-3

[33] A. Courand and M. Metz, "Evaluation of a robust regression method (roboost-plsr) to predict biochemical variables for agronomic applications: Case study of grape berry maturity monitoring," *Chemometrics and Intelligent Laboratory Systems*, vol. 221, 2022. [Online]. Available: https://doi.org/10.1016/j.chemolab.2021.104485

[34] C. Körner, "Mastering azure machine learning," in *Mastering Azure Machine Learning*. Packt Publishing, 2022.