Do Followers Really Matter in Stackelberg Competition?

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Abstract: *In this paper, we consider a T-stage linear model of Stackelberg oligopoly. First, we show geometrically and analytically that under the two conditions of linear market demand and identical constant marginal costs, the T-stage Stackelberg model reduces to a model where T oligopolies exploit residual demand sequentially. At any stage, leaders behave as if followers did not matter. Second, we study social welfare and convergence toward competitive equilibrium. Especially, we consider the velocity of convergence as the number of firms increases. The convergence is faster when reallocating firms from the most to the less populated cohort until equalizing the size of all cohorts.*

Keywords: *leader's markup discount ratio, linear economy, follower's output index, generalized Stackelberg competition. JEL classification: L13, L20.*

¿Importan realmente los seguidores en la competencia de Stackelberg?

Resumen: *En este artículo se considera un modelo de oligopolio de Stackelberg lineal en T etapas. En primer lugar, se muestra geométrica y analíticamente que bajo las condiciones de demanda de mercado lineal y costos marginales constantes e idénticos el modelo de Stackelberg en T etapas se reduce a un modelo en el que T firmas explotan la demanda residual secuencialmente. En cualquier etapa, los líderes se comportan como si los seguidores no importaran. En segundo lugar, se estudia el bienestar social y la convergencia hacia el equilibrio competitivo. En particular, se considera la velocidad de convergencia a medida que el número de firmas incrementa. La convergencia es más rápida cuando las firmas se relocalizan desde la cohorte más poblada a la menos poblada hasta que el tamaño de las cohortes se iguala.*

Palabras clave: *razón de descuento del markup del líder, economía lineal, índice de producto del seguidor, competencia de Stackelberg generalizada. Clasificación JEL: L13, L20.*

Les followers ont-ils vraiment de l'importance dans le modèle de Stackelberg?

Résumé : *Dans cet article nous considérons un modèle linéaire d'oligopole de Stackelberg avec T cohortes où les entreprises ont des stratégies en quantité. Tout d'abord, nous montrons géométriquement et analytiquement que, si la demande de marché est linéaire et les coûts marginaux sont constants et identiques, le modèle de Stackelberg à T étapes se réduit à un modèle où T oligopoles exploitent la demande résiduelle de manière séquentielle. À n'importe quelle étape, la stratégie des entreprises ne dépend ni du nombre d'entreprises qui jouent après, ni du nombre de cohortes restantes. Les entreprises leaders se comportent "comme si" les entreprises suiveuses n'avaient pas d'importance. Deuxièmement, nous étudions la convergence vers l'équilibre concurrentielle et le bien-être social. Nous considérons notamment la vitesse de convergence lorsque le nombre d'entreprises augmente.*

Mots-clés *: facteurs d'escompte markup, économie linéaire, modèle généralisé de Stackelberg. Classification JEL : L13, L20.*

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–Introduction. –I. The model. –II. Stackelberg competition in the linear economy. –III. Implications for convergence and welfare. –Conclusion. –Appendix. –References.

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Introduction

In the Von Stackelberg (1934) oligopoly model where firms interact in quantity, firms sequentially choose the quantities to produce and take into account the impact of their own decisions on the decisions of firms playing later. The basic model has notably been extended in order to integrate a larger number of stages and/or players than in the original model (Boyer and Moreaux, 1986; Sherali, 1984; Watt, 2002). An interest of such a structure, which is called a *hierarchy*,[1] is to introduce heterogeneity between firms according to their place in the decision process. Several implications have been derived concerning welfare (Watt, 2002), merging (Daughety, 1990; Heywood and McGinty, 2007, 2008) and profits (Etro, 2008), among others.

It has been stated by Boyer and Moreaux (1986), Anderson and Engers (1992) and Pal and Sarkar (2001) that under the two standard assumptions of linear market demand and identical constant marginal costs, a T-stage Stackelberg

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1 This denomination comes from Boyer and Moreaux (1986).

model reduces to a model where T monopolies exploit residual demand sequentially. Watt (2002) has extended the framework to integrate multiplayer cohorts.[2] These two assumptions constitute the so-called linear model.

The purpose of this paper is twofold. First, we geometrically derive and generalize the preceding statements, which are specific properties of the linear model. By contrast with the relevant literature, we provide an explanation for such a result by showing that leaders behave *as if* followers did not matter. The number of remaining stages and/or followers does not qualitatively modify the optimization program of a firm. A change in the number of stages and/or followers is embodied in a scale factor that homothetically discounts the objective profit function of the leaders, reducing profit without altering optimal strategies.

Second, we study the welfare implications of the linear model. Especially, we define a simple index of social welfare according to the number of firms and stages and thereby explore the convergence of the economy toward perfect competition. When the total number of firms in the economy or the number of firms in a given cohort becomes arbitrarily large, the T-stage Stackelberg equilibrium converges toward a competitive equilibrium. Furthermore, the convergence is faster when an additional firm enlarges the hierarchy rather than an existing cohort.

Our analysis is based on the existence of two scale factors: the *leader's markup discount ratio* and the *follower's output index*. The former represents the reduction of a leader's markup associated with the existence of his/her followers in the hierarchy. Its value differs from one cohort to another, depending negatively on the number of remaining cohorts and corresponding players. The latter represents the decrease in optimal quantities for a follower resulting from a contraction of the residual demand when playing latter in the hierarchy. It is a share of the first cohort production, whether optimal or not. For any follower, this share decreases when going further in the sequence. Its value depends negatively on the number of leading cohorts and on the number of corresponding players (because residual demand decreases with this parameter). Both factors measure the profit reduction of a firm within the hierarchy and are useful to analyze social welfare.

The paper is presented as follows. In section 1, we present the model. Section 2 analyzes the behavior of the firms under the Stackelberg structure,

2 Julien and Musy (2011) develop Watt (2002) by considering various degrees of competition between the followers.

and introduces the two kinds of discount factors. Section 3 derives welfare implications. In the last section, we conclude.

I. The Model

Consider one homogeneous good produced by n firms which oligopolistically compete in a hierarchical framework. There are T-stages of decisions indexed by $t \in [\![1,T]\!]$. Each stage embodies one cohort and is associated with a level of decision. The whole set of cohorts represents a hierarchy. Cohort t is populated by firms, with $\sum_t n_t = n$. The distribution of the firms within each cohort is assumed to be observable and exogenous.[3] This latter assumption notably implies that position of firms and timing of moves are given.[4]

A firm i which belongs to cohort t has to decide strategically (simultaneously with firms of the same cohort, and sequentially among the hierarchy) its level of output denoted by x_t^i. The aggregate output of cohort t is denoted $X_t = \sum_{i=1}^{n_t} x_t^i$, where x_t^i stands for firm i's output within cohort t. In addition, $X_t^{-i} = \sum_{-i} x_t^{-i}$ will denote the production of all firms belonging to cohort t but i.

We consider a linear Stackelberg model. Hence, the inverse market demand function, which specifies the market price p as a function of aggregate output X, with $X = \sum_{t=1}^{T} X$, is assumed to be $p(X) = a - bX$, $a, b > 0$. In addition, the cost function of any firm i which belongs to cohort t, is given by cx_t^i, $i = 1, \ldots, n_t$ and $t = 1, \ldots, T$. These two assumptions are standard in the literature on oligopoly analysis (see Daughety, 1990; Carlton and Perloff, 1994; Vives, 1999; among others).

The n_t firms which belong to cohort t, behave as followers with respect to all firms of cohort $\tau, \tau \in [\![1, t-1]\!]$, whose strategies are taken as given. However, they behave as Stackelberg leaders toward all firms of cohort $\tau, \tau \in [\![t+1, T]\!]$. They consider the best-response functions of all firms belonging to these cohorts as functions of their strategies. Therefore the profit of firm i which belongs to cohort t may be written:

$$\pi_t^i(x_t^i) = p\left(\sum_{\tau=1}^{t-1} X_\tau + \sum_{\tau=t}^{T} X_\tau\right) x_t^i - cx_t^i, \qquad i = 1, \ldots, n_t. \tag{1}$$

3 The standard Stackelberg duopoly prevails when T=2 and $n_1\, n_2 = 1$.

4 We therefore do not question the way a specific firm could or should become a leader (see Anderson and Engers, 1992; Amir and Grilo, 1999; Matsumura, 1999).

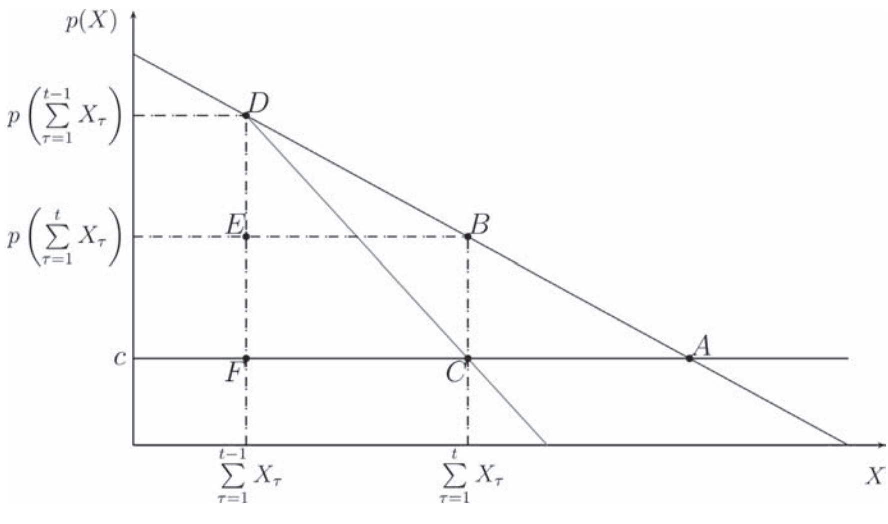
II. Stackelberg competition in the linear economy

A. Graphical interpretation

Consider two successive stages, say t-1 and t. Let $p\left(\sum_{\tau=1}^{t} X_\tau\right)$ be the market price when cohort t enters the market while each cohort $\tau(\tau<t)$ produces a quantity of output X_τ. We assume that any leading cohort $\tau<t$ expects firms of cohort t (or more) to act rationally and symmetrically. As in the standard literature, they maximize their profits for any quantity $\sum_{\tau=1}^{t-1} X_\tau$ produced by their predecessors. In this case, the rational choice of firms is depicted in Figure 1.

In this Figure, we illustrate the behavior of cohort t. Firms in cohort $t, t \in [\![1,T]\!]$, behave as Cournotian oligopolists on the residual demand left by firms of cohorts $\tau(1<\tau<t)$: it is as if they would not take into consideration firms playing after. So, the equilibrium strategies of firms in the hierarchical model coincide with those of a multistage Cournot model. Then, for every cohort, there is an equivalence between the sequential game and the (successive) static programs. In this paper, we enrich the meaning and implications of property 2, presented as the equivalence of the profit functions in the two models *up to a linear transformation*. This implies the equivalence of the reaction functions in both models and thereafter of the equilibrium strategies (which is then a consequence rather than a definition of the Cournotian behavior).

Figure 1: *Behavior of cohort t*



B. The equivalence between Stackelberg and Cournot behaviors

We now study formally the equivalence between the Stackelberg game and the successive Cournot games. It requires to exhibit the link between leaders and followers' profits.

Lemma 1 Let $\gamma_t \equiv \prod_{\tau=t+1}^{T} \frac{1}{1+n_\tau}$ be the leader's markup discount ratio. The markup earned by a cohort t firm, t < T, in a T-cohort economy is a constant share $\gamma_t < 1$ of the markup it earns in a t-cohort economy for any given vector of outputs $(X_1, ..., X_{t-1})$ produced by the previous cohorts:

$$p\left(\sum_{\tau=1}^{T} X_\tau\right) - c = \gamma_t \left[p\left(\sum_{\tau=1}^{t} X_\tau\right) - c \right] \quad \text{for t < T.} \tag{2}$$

Proof. See Appendix A.

Notice that under conditions on costs, the markup is always equal across cohorts. The discount factor γ_t measures the dependence of market power on the number of followers. It represents the reduction of markup of any leader due to the presence of the additional cohorts $t+1$ to *T*. It affects less intensively the market power of the last cohorts in the sequence since they face a reduced number of followers. Market power shrinks as *t* tends to infinity. This case will be discussed in Section 3.

The existence of cohort τ equally impacts by a coefficient $1/(1+n_\tau)$ the markup expected by a leader *t* ($t < \tau$) in a *t*-stage economy, whatever the quantities produced by the first *t* cohorts (this results directly from assumptions on demand and costs).

Corollary 1. For any strategy x_t^i, the profit obtained by a cohort-t firm in the sequential T-stage structure is a constant share of the profit earned in a t-stage economy:

$$\pi_t^i\left(x_t^i\right) = \left[p\left(\sum_{\tau=1}^{T} X_\tau\right) - c \right] x_t^i = \gamma_t \left[p\left(\sum_{\tau=1}^{t} X_\tau\right) - c \right] x_t^i . \tag{3}$$

Proof. This corollary directly results from Lemma 1.

In other words, each cohort can behave *as if* there were no following cohorts behind it since it earns a constant share of the profit realized in an oligopoly structure market where it represents the last cohort, whatever the aggregate output $\sum_{\tau=1}^{t-1} X_\tau$ produced by the leaders. Provided that cohort-*t* firms maximize their profit for any vector of strategies $(X_1, ..., X_{t-1})$, cohort-τ leaders ($\tau < t$) act as oligopolists ignoring the following cohorts. The existence of these additional cohorts does not distort fundamentally the maximization program, whose objective function (that is the profit function) is only discounted by a constant parameter. We have called this parameter the leader's markup discount ratio.

Lemma 2. Let $\eta_{t-h,t} \equiv \prod_{\tau=t-h+1}^{t} \frac{1}{1+n_t}$ be the follower's output index. In this economy, the output of a firm i in cohort t ≤ T can be expressed as a share of the output produced by a firm playing previously and belonging to cohort t − h for $h \in [\![1, t-1]\!]$, that is:

$$x_t = \eta_{t-h,t} x_{t-h} .$$

Proof. See Appendix B.

The follower's output index represents the contraction of output resulting from playing later in the hierarchy. It indicates the share of cohort *t-h*'s output which is optimal for cohort *t* to produce.

From Lemmas 1 and 2, the following proposition can be stated:

Proposition 1. When the market demand is linear and marginal costs are identical and constant, any cohort behaves as if followers did not matter. The T-stage Stackelberg linear economy reduces to a succession of staggered static problems in which firms compete oligopolistically on residual demands.

Proof. The proposition directly ensues from Lemmas 1 and 2.

Maximizing the right-hand side of equation (3) (sequential structure program) is tantamount to maximize the left-hand side of equation (3) since γ_t is a constant term. In the linear economy, strategies of firms do not depend on the number of firms playing after, which equally impact the profit associated to each strategy. As a consequence the optimal strategies and the equilibrium strategies remain unchanged whatever the number of stages and the number of followers in the sequential structure.

The literature only covers the similarity of the equilibrium strategies in both the *T*-stage Stackelberg linear model and the succession of staggered static problems but does not provide any explanation for this coincidence (see Boyer and Moreaux, 1986; Anderson and Engers, 1992; Watt, 2002).

Corollary 2 The equilibrium strategy of cohort 1-firms may thus be obtained from the profit maximization:

$$x_1 = \frac{1}{1+n_1}\frac{a-c}{b} \equiv \eta_1 X^*,$$

where $X^* = (a-c)/b$ is equal to the perfect competition aggregate output. We then deduce the equilibrium strategy of any firm i in cohort t, $t \in [\![1,T]\!]$:

$$x_t = \frac{\eta_{1,t}}{1+n_1}\frac{a-c}{b} \equiv \eta_t X^* \qquad \text{with} \qquad \eta_t \equiv \prod_{\tau=1}^{t}\frac{1}{1+n_\tau}.$$

Notice that in the equilibrium, each firm of cohort *t* produces a share η_t of the perfect competition equilibrium output.

Corollary 3 The equilibrium price and equilibrium profits are given by:

$$p = c + (a-c)\prod_{\tau=1}^{T}\frac{1}{1+n_\tau}$$

$$\pi_t^i = \frac{(a-c)^2}{b}\prod_{\tau=1}^{t}\frac{1}{(1+n_\tau)^2}\prod_{\tau=t+1}^{T}\frac{1}{1+n_\tau} \qquad t=1,\ldots,T$$

De Quinto and Watt (2003) use a similar term to η_t to analyze welfare through market power and mergers. In our approach, we investigate the issue of welfare through a comparison with perfect competition representing the maximizing global surplus benchmark case.

III. Implications for convergence and welfare

Social welfare is maximized under perfect competition, that is when aggregate output is equal to X^*. Let ω be the index of social welfare. This index, included between 0 and 1 (maximum welfare), is measured by the sum of the shares $n_\tau \eta_\tau$:

$$\omega = \sum_{\tau=1}^{T} n_\tau \eta_\tau = 1 - \frac{1}{(1+n_1)(1+n_2)\ldots(1+n_T)} = 1 + \eta_{1,T}$$

It can be asserted from Corollary 2 that the aggregate equilibrium output in the model is given by ωX^*.

Lemma 3 When the number of firms becomes arbitrarily large, either vertically (when T tends to infinity) or horizontally (when n_τ tends to infinity), the oligopoly equilibrium output converges toward the competitive equilibrium output.

Proof. Immediate from $\lim\limits_{T\to\infty}\sum_{\tau=1}^{T} n_\tau \eta_\tau = 1$ and $\lim\limits_{n_\tau\to\infty}\sum_{\tau=1}^{T} n_\tau \eta_\tau = 1$.

Convergence toward perfect competition is then achieved through an increase in the number of cohorts and/or in the number of firms in any cohort.

A specific case of vertical convergence can be found in Boyer and Moreaux (1986) for $n_\tau = 1$, $n_\tau = 1$, $t \in [\![1, T]\!]$.

From the previous lemma we know that welfare can be improved by increasing the number of firms. When the number of firms is fixed, welfare can be modified when firms are displaced in the decision sequence, either by enlarging the hierarchy or changing the size of existing cohorts.

Lemma 4 For any given number of firms, a displacement of any firm results in a higher welfare gain when enlarging the hierarchy rather than modifying the size of an existing cohort.

Proof. Assume a move of a cohort-*t* firm within the hierarchy. Let ω_1 be the social welfare index when this moves enlarges the hierarchy (adding a cohort T+1) and ω_2 be the same index when it modifies the size of an existing cohort (say t').

$$\omega_1 = 1 - \frac{1+n_t}{(1+n_t-1)(1+n_{T+1})} \prod_{\tau=1}^{T} \frac{1}{1+n_t} \quad \text{with} \quad n_{T+1} = 1,$$

$$\omega_2 = 1 - \frac{(1+n_t)(1+n_{t'})}{(1+n_t-1)(1+n_{t'}+1)} \prod_{\tau=1}^{T} \frac{1}{1+n_t}.$$

Since $\frac{1}{2+n} < \frac{1}{2}$ for any $n_t > 0$ then $\omega_1 > \omega_2$.

For a constant number of firms, adding new cohorts is always welfare improving. Said differently, introducing position-based asymmetries is welfare enhancing. It echoes and generalizes the result of Daughety (1990), which is restricted to T=2.

When both the number of stages and the number of firms are fixed, the following lemma shows how to improve welfare.

Lemma 5 For a fixed number of firms and cohorts, welfare improves as long as firms are relocated between cohorts until the difference of sizes between any two cohorts is at most equal to 1. For each relocation, welfare enhancement is greater when the firm is moved from the largest to the smallest cohort.

Proof. See Appendix C.

It can now be stressed the assumptions upon which positions of firms do not matter for social welfare, i.e. are invariant to specific modifications in the decision process. This property is called hierarchy neutrality.

Lemma 6 The linear economy is hierarchy neutral when relocation of firms consists of switching the whole cohorts within the hierarchy: this relocation does not affect social welfare.

Proof. Immediate: switching n_t and $n_{t'}$ backward or forward in $\eta_{1,T}$ does not change the value of ω.

From the preceding lemmas, one can state the following proposition relative to the link between welfare and the structure of the economy.

Proposition 2 In this linear economy, maximizing social welfare can be achieved through two ways:

(*i*) As a priority, by enlarging the hierarchy.

(*ii*) Then, by successively relocating firms from the most to the less populated cohort until equalizing the size of all cohorts.

Proof. Proposition 2 ensues from Lemmas 4 to 7.

This proposition could also be used to analyze how entry affects welfare. If new firms enter the economy, the increase in welfare is greater if new cohorts are created rather than if those firms integrate existing cohorts.

Conclusion

The paper investigates a hierarchic T-stage oligopoly model. It states that followers do not matter in the linear case, i.e. under constant identical marginal costs and linear demand. This means that at any stage each firm behaves as a Cournotian oligopolist on residual demand. In addition, the two discount factors presented in this paper enable us to characterize to fully characterize the market outcome of the linear economy, especially in terms of strategies and welfare.

In Julien, Musy and Saïdi (2011), we show that this property holds in the linear economy exclusively, provided the marginal cost is strictly positive. Once one of the linear assumptions is relaxed, the results disappear. However, the linear economy is a useful benchmark to determine the optimal strategies of firms in more general and complex economies.

Appendix A. Proof of Lemma 1

The proof is by backward induction and structured in three steps.

<u>Step 1</u>: assume equation (2) is true for cohort $t = T-1$ (with $T > 1$).

The inverse demand function faced by firms (blue line) is defined by:

$$p(X) = a - bX \quad \text{with} \quad X = \sum_{\tau=1}^{T} X_\tau \qquad \text{(H1)}$$

where $X_\tau = \sum_{i=1}^{n_\tau} x_\tau^i \geq$ is the aggregate production of cohort τ. For any quantity of output X_{T-1} produced by cohort $T-1$, the resulting residual demand faced by followers of cohort T is:

$$p\left(\sum_{\tau=1}^{T} X_\tau\right) = \hat{a}_{T-1} - bX_T \quad \text{with} \quad \hat{a}_t \equiv a - b\sum_{\tau=1}^{t} X_\tau \ ,$$

where $\hat{a}_{T-1}$ is considered as given by followers. Geometrically, followers must select a couple (X, p) on the segment [D, A].

When acting symmetrically, the associated marginal revenue of cohort-T firms (red line) is defined by:[5]

$$R_m\left(X_T\right) = \hat{a}_{T-1} - b\frac{1+n_T}{n_T}X_T \ .$$

Considering the following derivatives:

$$\frac{\partial R_m}{\partial X_T}\left(X_T\right) = \frac{DF}{CF} = -b\frac{1+n_T}{n_T}$$

$$\frac{\partial p}{\partial X_T}\left(X_T\right) = \frac{DF}{AF} = -b \ ,$$

it comes that:

$$CF = \frac{n_T}{1+n_T}AF \ , \ \text{or equivalently} \ \ AC = \frac{1}{1+n_T}AF \ .$$

Finally, applying Thales' theorem to triangles ABC and ADF leads to:

$$BC = \frac{1}{1+n_T}DF = EF \ . \qquad \text{(A.1)}$$

Actually, EF is the markup of a leader after the entrance of the last cohort, while DF is the markup of a leader before the entrance of cohort T. Equation (A.1) can be rewritten as:

$$p\left(\sum_{\tau=1}^{T} X_\tau\right) - c = \frac{1}{1+n_T}\left[p\left(\sum_{\tau=1}^{T-1} X_\tau\right) - c\right]$$

---

5 This function is derived from the total revenue of a follower i: $RT\left(x_T^i\right) = \left[\hat{a}_{T-1} - b\sum_{k=1}^{n_T} x_T^k\right]x_T^i$. The symmetric behavior assumed for followers yields: $x_T^i = x_T$ for all $i \in \left[1, n_T\right]$ and $X_T = n_T x_T$.

<u>Step 2</u>: assume equation (2) is true for any cohort $t = T - h$ $(1 \leq h \leq T-2)$ then it is true for cohort $T - h - 1$.

If equation (2) holds for cohort $T - h$ then:

$$\left[p\left(\sum_{\tau=1}^{T} X_\tau\right)-c\right]x_{T-h}^{i} = \gamma_{T-h}\left[p\left(\sum_{\tau=1}^{T-h} X_\tau\right)-c\right]x_{T-h}^{i} \quad \text{with} \quad \gamma_{T-h} \equiv \prod_{\tau=T-h+1}^{T}\frac{1}{1+n_\tau}$$

Thus, maximizing firm i's profit is tantamount to maximize the $T-h$-stage profit defined as follows:

$$\max_{x_{T-h}^{i}}\left[p\left(\sum_{\tau=1}^{T-h} X_\tau\right)-c\right]x_{T-h}^{i}.$$

When firms of cohort $T-h$ act symmetrically, the corresponding marginal revenue (red line) is defined by:[6]

$$\tilde{R}_m\left(X_{T-h}\right) = \hat{a}_{T-h-1} - b\frac{1+n_{T-h}}{n_{T-h}}X_{T-h}.$$

In the same way as in step 1, it can be shown that:

$$p\left(\sum_{\tau=1}^{T-h} X_\tau\right)-c = \frac{1}{1+n_{T-h}}\left[p\left(\sum_{\tau=1}^{T-h-1} X_\tau\right)-c\right].$$

By assumption, the following property is satisfied:

$$p\left(\sum_{\tau=1}^{T} X_\tau\right)-c = \gamma_{T-h}\left[p\left(\sum_{\tau=1}^{T-h} X_\tau\right)-c\right].$$

We deduce from the two previous equations that:

$$p\left(\sum_{\tau=1}^{T} X_\tau\right)-c = \frac{\gamma_{T-h}}{1+n_{T-h}}\left[p\left(\sum_{\tau=1}^{T-h-1} X_\tau\right)-c\right]$$

$$= \gamma_{T-h-1}\left[p\left(\sum_{\tau=1}^{T-h-1} X_\tau\right)-c\right].$$

6 The associated marginal revenue is: $R_m\left(X_{T-h}\right) = \tilde{R}_m\left(X_{T-h}\right) + \left(1-\gamma_{T-h}\right)c$.

Step 3: from steps 1 and 2 we conclude by backward induction that equation (2) is true for any cohort t (with $1 \leq t \leq T-1$).

Appendix B. Proof of Lemma 2

Applying Thales' theorem to triangles ABC and ADF leads to:

$$CF = \frac{n_t}{1+n_t} AF. \qquad \text{(B.1)}$$

Actually, CF is the optimal output produced by cohort t, that is X_t, while AF is the maximal quantities cohort t can produce to generate non-negative profit (equal to the difference between the perfect competition equilibrium supply and the output already produced by the previous cohorts). The property above can be rewritten as:

$$X_t = \frac{n_t}{1+n_t}\left(X_t + AC\right), \text{ or equivalently } \quad AC = \frac{X_t}{n_t} = x_t.$$

Notice that AC is also the maximal quantities cohort $t+1$ can produce to generate non-negative profit. Then, equation (B.1) applied to cohorts t and $t+1$ becomes:

$$X_{t+1} = \frac{n_{t+1}}{1+n_{t+1}} AC, \quad \text{leading to} \quad \frac{X_{t+1}}{n_{t+1}} = x_{t+1} = \frac{1}{1+n_{t+1}} x_t.$$

By backward induction, it turns out that:

$$x_t = \eta_{1,t} x_1, \text{ where } \quad \eta_{1,t} \equiv \prod_{\tau=2}^{t} \frac{1}{1+n_\tau}.$$

Appendix C. Proof of Lemma 6

Maximizing the welfare index ω is tantamount to maximize:

$$\begin{cases} \max\limits_{n_\tau} \prod\limits_{\tau=1}^{T} 1+n_\tau \\ s.t. \sum\limits_{\tau=1}^{T} n_t = n \end{cases}$$

Substituting n_T by $n - \sum_{\tau=1}^{T-1} n_\tau$ into the objective function, deriving with respect to n_τ, $\tau \in [\![1, T-1]\!]$, and assuming the $n_\tau' s$ are infinitely divisible yields the following first-order conditions:

$$n-\sum_{\tau=1}^{T-1} n_\tau = n_t, \quad t \in [\![1, n_{T-1}]\!],$$

or equivalently (in addition with the definition of n_T):

$$n_t = n/T \equiv \bar{n}, \ t \in [\![1, n_{T-1}]\!] .$$

Notice that for this value of n_τ the omitted constraint $0 \le n_\tau \le n$ is satisfied for any $\tau \in [\![1, T]\!]$.

At $(\bar{n},...,\bar{n})$, the $T \times T$ Hessian matrix $\mathbf{M}$ is

$$\mathbf{M} = -(1+\bar{n})^{T-2}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} - (1+\bar{n})^{T-2} I,$$

where I is the identity matrix. The eigenvalues of $\mathbf{M}$ are $-(T+1)(1+\bar{n})^{T-2}$ and $-(1+\bar{n})^{T-2}$ (the associated eigenspaces have dimension 1 and $T-1$ respectively).

Matrix $\mathbf{M}$ is then negative definite:

- The unique solution to the first-order conditions is a global maximum when n/T is an integer.
- There are multiple optima when n/T is not an integer. Due to the strict concavity of the objective function, these optima must be as close as possible to the hypothetical solution above. In other words they must minimize the distances $\left|n_\tau - \frac{n}{T}\right|$, for $\tau = 1,...,T$, such that $\sum_{\tau=1}^{T} n_\tau = n$. The minimum value of these distances is 1 and can be obtained as follows.

  Let $m < T$ be an integer such that $(n-m)/T = \lfloor n/T \rfloor$. An optimum is such that there are $T-m$ cohorts populated by $\lfloor n/T \rfloor$ firms and the other m cohorts by $\lfloor n/T \rfloor + 1$ firms. The number of combinations of m cohorts out of T defines the number of optima.

  Notice that the most populated cohorts embody one more firm than the less populated cohorts.

When n/T is not an integer and for given values of the $n_\tau' s$, the more efficient way to get closer to the hypothetical optimal as one firm is relocated consists in reducing the largest distance, e.g. $\left|n_t - \frac{n}{T}\right|$. Without loss of generality, assume that the difference $\left(n_t - \frac{n}{T}\right)$ is positive. Then, it is not efficient for social welfare to relocate the firm in a cohort t' with $n_{t'} > \bar{n}$ since this move decreases $\left|n_t - \frac{n}{T}\right|$ but increases $\left|n_{t'} - \frac{n}{T}\right|$. The firm must be relocated in a cohort τ such that $n_\tau < \bar{n}$. Within this set of cohorts, the cohort with the largest $\left|n_\tau - \frac{n}{T}\right|$ will be selected since the reduction of the distance is the highest.

The argument is similar when the difference $\left(n_t - \frac{n}{T}\right)$ is negative. As a conclusion, social welfare is more efficiently improved when relocating a firm from the most to the less populated cohort.

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