Calculation of the Production Function through Entropy: A Model from Econophysics

Isabel Cristina Betancur-Hinestroza, Efraín Arango-Sánchez, Álvaro Hernán Bedoya-Calle, Francisco J. Caro-Lopera and Éver Alberto Velásquez Sierra

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Abstract: *From an economic perspective, the problems associated with production require a model for its estimation. These estimations are typically made using the Cobb-Douglas production function, which is derived from least-squares adjustment. However, several authors have proposed alternative methods for calculating production based on the relationship between economic and thermodynamic problems. Our study provides a method for calculating the production function through entropy, which explicitly considers the contributions of labor and capital. In this way, we apply the model to the data presented in the Cobb-Douglas study on the production of the manufacturing sector in the United States. Our results accurately describe the production data, highlight the need to estimate the Boltzmann constant in the economic model, and provide a method for determining its value.*

Keywords: *Econophysics, Economic Entropy, Cobb-Douglas function, production, Microeconomics.*
JEL Classification: A12, D24, C02.

Cálculo de la función de producción mediante entropía: un modelo desde la econofísica

Resumen: *Desde una perspectiva económica, los problemas asociados con la producción requieren un modelo para estimarla. La forma más habitual de hacer estas estimaciones ha sido tradicionalmente a través de la función de producción Cobb-Douglas, que proviene del ajuste por mínimos cuadrados. Sin embargo, varios autores han propuesto formas alternativas de calcular la producción basándose en la relación entre los problemas económicos y los problemas termodinámicos. Nuestro artículo muestra una forma de calcular la función de producción a través de la entropía, la cual considera explícitamente las contribuciones del trabajo y el capital en la entropía de la función de producción. De esta manera, aplicamos el modelo a los datos presentados en el estudio de Cobb-Douglas sobre la producción del sector manufacturero en Estados Unidos. Los resultados describen los datos de producción en un grado adecuado. Además, indican la necesidad de estimar la constante de Boltzmann en el modelo económico y presentan una propuesta para obtener su valor.*

Palabras clave: *econofísica, entropía económica, función Cobb-Douglas, producción, microeconomía.*

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Calcul de la fonction de production à l'aide de l'entropie: un modèle issu de l'éconophysique

Résumé: *D'un point de vue économique, les problèmes associés à la production nécessitent un modèle pour l'estimer. La façon la plus courante d'effectuer ces estimations est traditionnellement la fonction de production Cobb-Douglas, issue de l'ajustement des moindres carrés. Cependant, plusieurs auteurs ont proposé d'autres façons d'estimer la production en se basant sur la relation entre les problèmes économiques et les problèmes thermodynamiques. Notre article montre une façon de calculer la fonction de production à travers l'entropie, qui considère explicitement les contributions du travail et du capital dans l'entropie de la fonction de production. Nous appliquons ainsi le modèle aux données présentées dans l'étude Cobb-Douglas de la production manufacturière aux États-Unis. Les résultats décrivent parfaitement les données de production. En outre, ils indiquent la nécessité d'estimer la constante de Boltzmann dans le modèle économique et présentent une proposition pour obtenir sa valeur.*

Mots clés: *econophysique, entropie économique, fonction Cobb-Douglas, production, microéconomie.*

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Calculation of the Production Function through Entropy: A Model from Econophysics

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–Introduction. –I. Theoretical Framework. –II. Methodology. –III. Results. –IV. Finding -Conclusions. –Acknowledgment. –Conflicts of Interest. –Data Availability Statement. –Ethics Statement. –References.

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Introduction

An essential part of the study of economics, especially microeconomic theory, is concerned with the theory of production or the theory of the firm. This explains the fundamental relationships among allocation, welfare, economic benefits, and efficiency. One of the primary challenges faced by a firm is the decision to maximize the number of units produced using the available resources; this task is referred to as the first production problem. Conversely, the company may aim to achieve a target of units produced at

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the lowest possible cost, a challenge usually known as a dual production problem (Jehle & Reny, 2001). In very general terms, firms demand factors of production and combine them to obtain a final product, which is subsequently sold in a market to generate economic benefits. In both scenarios, the company will seek to determine the optimal use of the production factors, with classical problems involving only two of these factors: capital and labor. Although total production is considered a dimensionless variable, it is typically measured in machine-hours and man-hours. The well-known Cobb-Douglas production function is a widely used method to calculate this production problem (Pindyck & Rubinfeld, 2017; Varian, 2014). It is a reasonable model that is adjusted to the expected characteristics and properties of the economic rationality assumption. Estimating the parameters of the production function from the capital and labor data is crucial, as this function often links the scale of technology and the elasticities of production to changes or variations in the labor and capital factors. For instance, one study employs the Cobb-Douglas function to quantify the influence of training on business productivity within the sustainable public service sector in Europe during the period from 2008 to 2010 (Pedrini & Cappiello, 2022). Another study utilizes a similar function to assess the effects of job training on enhancing productivity and wages across 170,000 companies in Belgium (Konings & Vanormelingen, 2015).

Recent research has unveiled diverse applications of the Cobb-Douglas function. Adekanye and Oni (2022) explored the significance of energy utilization and its impact on cassava production in Nigeria, employing this function to assess the influence of inputs on overall cassava yield. Another study assessed economic efficiency across 14 countries in Latin America and the Caribbean from 1990 to 2007 through the Cobb-Douglas function to elucidate the concept of efficiency in maximizing production with a specific set of inputs (Koengkan *et al.,* 2022). Similarly, further research investigated green mussel cultivation on Pasaran Island, Lampung Province, Indonesia (Sulvina *et al.,* 2020), using multiple linear regression analysis with the Cobb-Douglas function to determine the factors impacting production and efficiency in tool and material utilization. This study also determined the efficiency levels of various production factors in mollusk farming, demonstrating increasing scale returns within the production function. Additional research examined the contribution of human capital to the agricultural production process in Burkina Faso, West Africa, by applying an augmented Cobb-Douglas agricultural production function (Wouterse & Badiane, 2019). The Cobb-Douglas function was also applied to gauge technical efficiency in potato farming, underscoring the positive impact of socio-economic variables on technical efficiency and identifying the distance from farms to markets as a contributor to technical inefficiency (Wassihun, 2019).

On the other hand, since the 1990s, an alternative proposal has been presented by econophysics to calculate the production function using entropy. Entropy establishes connections between physical and economic systems by employing the calculus of differential functions and the laws of thermodynamics. This highlights the advantages of economic entropy in relation to the production function (Mimkes, 2006; Richmond *et al.,* 2013). Giuseppe Palomba was the first to establish the similarities between physics and economics in his book "Economic Physics" (Gallegati, 2016). Georgescu-Roegen (1986) later illustrated several relationships between these two areas and explicitly demonstrated thermodynamics relationships through statistical physics and economic processes. This author identifies a close relationship between the second law of thermodynamics and economic processes.

Stanley coined the term "Econophysics", referring to the branch of complex physics systems that sought to characterize the statistical properties of a financial market with non-preferred probability distribution functions (Stanley *et al.,* 1996). Hence, probabilistic models should be robustly estimated from the available data. This perspective has generated controversy since traditional economic assumptions are predominantly based on the normal distribution. Nevertheless, these contributions demonstrate how economic concepts can be built on the principles of physics, yielding more intuitive results for understanding economic phenomena.

Other findings have revealed the historical contributions of physics to the economy, such as a review of Adam Smith's motivation for applying econophysics as a novel discipline that provides instruments for investigating economics, grounded in the ideas of Isaac Newton (Pereira *et al.*, 2017). Another study examines the role of thermodynamics, with a particular focus on entropy, in the context of economic studies over the past 150 years (Jakimowicz, 2020). Other applications develop links between the Law of Diminishing Marginal Utility, the thermodynamics expression of the marginal entropy value, and the Cobb-Douglas function (Bryant, 2007). Based on the Joule cycle from physics, the latter work developed a commercial cycle using equations of efficiency, growth, and entropy gain. It proposed a monetary model that is analogous to thermodynamic concepts applied to ideal gases and is based on the economic properties of the monetary system. Similarly, a further investigation models the entropy production function based on a profit maximization function, allowing larger flexibility and heterogeneity in input substitutability. The article emphasizes that the entropy production function can establish a connection between microeconomic theory and entropy/information theory models used in price equilibrium modeling (Miyagi, 1994).

The literature review reveals a number of econophysics applications that establish conceptual comparisons and conduct applied exercises. For instance, one study presents an alternative for the dynamic management of income for product pricing that considers the errors found in the optimization processes of this type of analysis; this alternative represents the uncertainty in the interest rate demand model with relative entropy concerning the nominal probability (Lim & Shanthikumar, 2007). An additional study analyzes a dynamic duopoly game represented by a version of the Cobb-Douglas function (Askar & Khedhairi, 2020). This work uses entropy as a statistical measure to determine the consistency of the model in terms of its regularity and predictability in time series. Tsigaris and Wood (2016) conduct an exercise on economic growth and entropy, and Backus *et al.* (2014) develop a financial application using this same function.

Other investigations employ the principle of maximum entropy for specific applications. This principle proposes assigning utility values with partial information for decision-making (Abbas, 2006). Using data from some agricultural sectors in México, Howitt and Msangi (Howitt & Msangi, 2014) propose a robust approach to estimating maximum entropy production functions capable of reproducing characteristics and predicting the outcomes of policy changes. Similarly, the maximum likelihood method is substituted by the maximum entropy method to estimate rice production in a province in Thailand (Autchariyapanitkul *et al.,* 2017). This method has been used to assess the impact of digital technologies on the Thai economy (Chakpitak *et al,* 2018) and to demonstrate the link between non-energy demand for renewable energy and economic growth in Bangladesh (Alam *et al.,* 2017). Faced with the difficulty of characterizing the probability distribution of changing demand, Perakis and Roles (2008) study the newsvendor problem with partial information about demand distribution, noting that distributions that maximize entropy perform well for specific criteria.

The above-mentioned works have demonstrated the various applications and developments that support the use of entropy in the study of economic matters. However, the implementation of entropy for computing the production function in the setting of a fundamental economic problem has not been researched. In this sense, some theoretical studies have presented arguments for calculating the production function through economic entropy instead of the Cobb-Douglas proposal (Mimkes, 2006; Richmond *et al.*, 2013). Initially, they argue that the production function calculated through entropy does not require estimating parameters such as elasticities α and β, as it does not stem from an adjustment process but rather from the relationship between the variables and laws of the economy and the quantities and laws of thermodynamics. Furthermore, the application results in optimization problems demonstrate that entropy generates higher production values and lower costs than those obtained using the Cobb-Douglas function. However, the referred authors do not test their proposal to calculate output through entropy in an entire economic system. In addition, they assume a value of one for the constant required in the calculation of entropy (Boltzmann's constant in the economic model). In the framework of the defined thermodynamics, a well-established value for the Boltzmann constant shows the connection between the system's entropy or disorder and the relationship between internal energy and temperature. In this study, we employ the entropy framework established by Richmond *et al.* (2013) and Mimkes (2006) to calculate the production function, considering the explicit contributions of labor and capital to entropy. Our findings highlight the importance of estimating the value of the corresponding econometric Boltzmann constant for the economic model and present a proposal for conducting this estimation. The fundamental goal of research in both fields is to broaden the interpretative scope of the economic phenomenon or to transition certain concepts that might remain obfuscated to our analytical scrutiny if approached differently. An illustrative case of this endeavor involves elucidating the equilibrium between orderliness and chaos within production processes, advocating for a dynamic paradigm to apprehend economic efficiency, predicated upon empirical evidence derived from real-world production activities.

In this setting, the paper is organized as follows: Section I provides an econophysics theoretical framework by proving the relationship between thermodynamic and economic variables. Section II explains the general methodology for computing economic entropy via the Stirling approximation. Section III subsequently applies and thoroughly discusses the research methods in relation to a well-known production dataset. The paper ends with some conclusions and directions for future research.

I. Theoretical Framework

The production function has been extensively calculated in the last century, both in the aforementioned research contributions and as a reference in the majority of economic theory textbooks (Pindyck & Rubinfeld, 2017; Varian, 2014). Accordingly, based on the dominant theory of the neoclassical economic model, the most widely used expression to represent a production function (Q) in two-factor systems has been calculated through the Cobb-Douglas function:

$$Q\left(L,K\right)=\mathrm{A}L^{\alpha}K^{\beta} \tag{1}$$

where L corresponds to work or labor measured as the average number of employees and K denotes capital measured in monetary terms or in physical units, such as the amount of equipment or investment in infrastructure.

According to the exercise conducted by Cobb-Douglas (1928), Q represents the average production of companies in the manufacturing sector in the United States during the analysis period, while α and β are the results of a statistical estimation using ordinary least squares to find the best-fit curve to the observed data set as well as a mathematical function for predicting future values. In this context, the values of α and β allow for measuring the impact of changes in capital and labor on production and, consequently, their productivity in the production system, as each factor does not necessarily contribute equally to the volume of production. At this stage, it must be noted that Cobb-Douglas (1928) and subsequent related works do not perform any statistical analysis based on the Gauss-Markov theorem for explaining the significant optimal required solutions. We will revisit this problem in Section III.

Similarly, α and β represent the elasticities of production in the face of changes in labor and capital factors, respectively, while A denotes the total productivity factor. Conversely, some authors define it as the factor productivity scale. It is typically a non-factor variable that incorporates the company's organization and culture, the trajectory and human capital of employees and management, and the implementation of technological advancements (Solow, 1957).

Cobb-Douglas (1928) proposed an alternative production function that is derived from physics and the calculus of functions of two variables and is based on the relationship between economic systems and many-particle systems in statistical physics (thermodynamics). This proposal offers an alternative approach for calculating the production function through entropy (Mimkes, 2006; Richmond *et al.,* 2013). In this respect, quantities in both thermodynamics and economics can only be calculated once the process is complete. Future earnings or rates of return cannot be predicted a priori by the economic context. Moreover, as in physics, the energy invested per unit of work, or the heat absorbed or given up during a process, cannot be calculated since both quantities depend on the state's final and initial trajectory. In two-dimensional differential calculus, these types of quantities are known as inexact differentials. They are the basis for relating the thermodynamic variables of internal energy, work, heat, and entropy to economic variables, such as capital, labor, profit (or loss), and the production function, respectively (Mimkes, 2019).

A. Relationship between Thermodynamic and Economic Variables

From a thermodynamic perspective, let us consider a closed system, defined as one that does not exchange particles with its surroundings while still interacting with them. The change in its energy, ΔE, results from two contributions: work (W) and heat (Q). These quantities are interconnected through the well-known first law of thermodynamics, or the law of energy conservation:

$$\Delta E = Q + W \tag{2}$$

In terms of work, the energy of the system decreases when it performs work, considering that any system must invest a portion of its energy to perform work; similarly, when work is performed on the system, its internal energy increases. The case for heat is analogous: when a system is in thermal contact with its surroundings, it can exchange energy in the form of heat, releasing heat when its surroundings are at a lower temperature than the system and, conversely, absorbing heat when the surroundings are at a higher temperature.

Equation (2) can be expressed in differential form as follows:

$$dE = \delta Q + \delta W \quad \text{or} \quad \delta Q = dE - \delta W \tag{3}$$

Energy is an exact differential form, as the change in total energy depends only on the initial and final states, meaning it is independent of the path taken. The change in energy remains the same if the last and initial states are maintained. This is not the case with the quantities of work and heat separately, as these depend on the initial and final states and the path taken. Thus, they can only be determined once the process is complete, as the path they will follow cannot be known a priori. Therefore, both quantities are inexact differential forms (Adkins *et al.,* 1983).

Conversely, when a cycle is carried out on a system—a process in which the final and initial states are the same—the energy does not change, but heat and work do. This can be expressed mathematically in the following way, in which case:

$$\oint dE = 0 \tag{4}$$

$$\oint dE = \oint \delta Q - \oint \delta W = 0 \tag{5}$$

This implies that:

$$\oint \delta Q = \oint \delta W \tag{6}$$

indicating that work and heat in a cycle are non-zero, but the change in energy is zero.

Similarly, discussing economic circuits is essential to understanding market dynamics and the relationship between thermodynamics and economics. Monetary and productive circuits are based on the existence of two economic agents, households and industry, as well as the relationship between them. The productive circuit is based on the work (W) performed by households (N) (belonging to the social system) in industries (K), after which industries or companies (belonging to the commercial system) sell goods (G) to households through consumption. Simultaneously, the monetary circuit (M), which quantifies production (P), is implicit in this transaction, implying that the industry pays wages (YH) for the work performed (W), and households pay the cost (CH) for the goods (G) they purchase. This monetary circuit values all its actions in monetary units, i.e., wages YH and costs CH.

There can be surpluses or losses for companies and households in the monetary circuit. For companies, surpluses occur when money from goods and services exceeds production costs (surplus or profits). For consumers, surpluses arise when households spend less than they earn. Conversely, companies and households will face losses when the opposite occurs. Nevertheless, it is impossible to predict the value of the surplus or deficit for every process in each circuit at any given time. This implies that the relationship between these variables can be expressed in inexact differential forms analogous to the heat and work examples presented earlier.

The first law of economics, as referred to by some authors in econophysics, is derived from the aforementioned elements (Chakrabarti *et al.,* 2006; Mimkes, 2019; Richmond *et al.,* 2013). As described earlier, production in any sector involves the use of labor and the transformation of production factors in a cyclic manner. However, production depends on how these factors are utilized. Since production cannot be calculated ex-ante, in mathematical terms, it can be considered an inexact differential form. $\oint \delta P$ expresses the cycle of goods or production; likewise, the production volume cannot be determined ex-ante. Similarly, the gain $\oint \delta M$ cannot be obtained a priori. These two cycles go in opposite directions, and their relationship is expressed by:

$$\oint \delta M = - \oint \delta P \tag{7}$$

This relationship is similar to equation (6), whereby (δM) and ($-\delta P$) are expressed as exact differentials by introducing a total differential. While (δM) and ($-\delta P$) are equal along the integration path, they can only differ by a total differential form, denoted as dK. Since the closed integral of dK is zero, the surplus in production (δM) can only be generated by work ($-\delta P$), not by capital (dK). In other words, this first law or financial balance, in which the gain (δM) obtained by a system such as a company or a household is determined by variations in capital (dK) and what is invested in labor or work (δP), is expressed as follows:

$$\delta M = dK - \delta P \tag{8}$$

Indeed, the terms of this equation are clearly measured in monetary units, where δW represents the investment made for the payment of workers or the execution of tasks leading to a gain δM. The first law of economics posits that the surplus from production (δM) could augment capital (dK) through investment in or utilization of work ($-\delta P$). In other words, for the surplus to contribute to capital growth, the use of labor and resources in the production process is imperative.

Drawing an analogy from the previously described first law of thermodynamics, relationships are established between energy (dE) and capital (dK), work (δW) and the effort exerted in labor (δP), and finally, heat (δQ) and gains or losses δM.

Table 1 compares thermodynamic and econophysical concepts based on equation (8) and the variables involved in the first law of econophysics. Analogous relationships between quantities can be established; however, it is noteworthy that in econophysics, work measured in monetary units is always negative, as it represents an expenditure or investment necessary to carry out the production process. In contrast, in thermodynamics, this quantity can be positive or negative depending on the change in the system's volume.

B. Relationship between the Second Law of Thermodynamics and the Second Law of Econophysics

Concerning calculus principles, an inexact differential form such as δQ can be related to an exact differential via an integrating factor. This leads to $dS = \delta Q/T$, where T is temperature and dS is entropy. In thermodynamics, systems in equilibrium are characterized by their temperature (T).

Rewriting equation (3) in terms of entropy as an exact differential yields:

$$dE = \delta Q + \delta W = TdS + \delta W \tag{9}$$

Considering a system with constant N (number of particles) and V (volume), work is constant; therefore:

$$dE = \delta Q = TdS \tag{10}$$

In this manner, it can be deduced that:

$$\left[\frac{dE}{dS}\right]_{V=cte} = T \tag{11}$$

The goal at this point is to obtain an expression for entropy, reorganizing:

$$\frac{dS}{dE} = \frac{1}{T} \tag{12}$$

Table 1. *Conceptual Relationship between Thermodynamic Equations and Econophysics*

Physics	Econophysics
$dE = \delta Q + \delta W$	$dK = \delta M + \delta P$
The first law of thermodynamics assesses how changes in internal energy result from transfers of heat and/or work.	The first law of economics evaluates changes in total capital from money invested in labor and the surplus generated by a transaction of goods in the market.
dE: Energy	**dK: Kapital**
The change in internal energy occurs through a temperature change. If the temperature remains constant, the system's energy does not change.	This variable represents the change in capital or total resources caused by variations in firm profitability.
δQ: Heat	**δM: Profits or losses**
Heat is positive when the system is in contact with surroundings that have a higher temperature and, therefore, energy is transferred to the system by heat. Conversely, when the surroundings exhibit a lower temperature, the system transfers energy to them through heat, and in this way, the heat is negative.	Depending on the market and the business conducted, profits ($\delta > 0$) or losses ($\delta M < 0$) can be generated after the production process.
δW: Work	**δP: Labor or productivity**
Work is positive when it is done on the system, in which case the system decreases in volume and increases its energy due to work. Conversely, work is negative when the system performs work, causing the system to expand and decrease its energy.	Negative work implies that there is an expenditure of capital to carry out the production process.

Source: Own elaboration based on Mimkes (2019), Chakrabarti et al. (2006) and Richmond et al., (2013).

Moreover, the energy of a system, E, is the sum of the energies possessed by all its constituent particles. However, there are numerous ways in which the particles can be arranged to reach this energy value. This quantity, denoted by Ω, represents the number of possible microstates that can yield the given energy.

According to Beale and Pathria (2021), for systems in thermal equilibrium, a relationship between temperature and Ω can be derived:

$$\beta = \frac{d \ln \Omega}{dE} \tag{13}$$

where $\beta = \frac{1}{k_B T}$

Through a comparison between equations (12) and (13), we find that:

$$\Delta S = \frac{1}{\beta T} \Delta(\ln \Omega) \tag{14}$$

Therefore, the entropy:

$$S = \frac{1}{\beta T} \ln \Omega = k_B \ln \Omega \tag{15}$$

In this expression, k_B is the Boltzmann constant, which has the same units as the entropy and links this quantity to the system's disorder, as determined by the value of Ω. Additionally, k_B links the temperature and the internal energy of an ideal system (ideal gas) that is in thermal equilibrium. This equation also indicates that, in an isolated system in thermal equilibrium, there is no a priori reason for one microstate to be preferred over another.

In the same manner as physical quantities were related to economic quantities, we can propose a relationship for economic entropy through equation (15). It is also suggested that this economic entropy (S_E) can be considered to calculate the production function in a manner similar to Richmond *et al* (2013) and Mimkes (2006):

$$S_E = A_E \times \ln(\Omega) \tag{16}$$

Here, A_E is similar to the Boltzmann constant for the economic model, which we will name the economic Boltzmann constant. This constant must have a value of one. However, we will estimate this value for the economic model of the problem to be considered (Richmond *et al,* 2013 and Mimkes, 2006).

As already mentioned, Ω in equation (16) is the number of different ways of distributing N elements in k different categories. For example, in the case of a two-category system such as a company with N_1 number of qualified employees and N_2 of unqualified ones (Richmond *et al,* 2013 and Mimkes, 2006), it would be:

$$\Omega = \frac{N!}{N_1!N_2!} \tag{17}$$

where N is known, and the total number of elements is $N = N_1 + N_2$.

For this type of problem, N is related to the total number of employees in the company. However, it can also be considered as the total number of elements or production factors that contribute to the production process. Note that Ω is the number of different ways of achieving the same microstate, so one approach is to consider alternative methods to distribute capital through various contributions. Therefore, to generate a certain production process, contributions are needed from both the amounts invested by labor (L) and capital (K). In this way, we incorporate these contributions to the production process into our model. Hence, the equation (17) can be written as:

$$\Omega = \frac{(L+K)!}{L!K!} \tag{18}$$

Thus, the economic entropy is obtained as:

$$S_E = A_E \times \ln\left[\frac{(L+K)!}{L!K!}\right] \tag{19}$$

Furthermore, using the properties of logarithms, it can be written as:

$$S_E = A_E \left[\ln\left((L+K)!\right) - \ln\left(L!\right) - \ln(K!)\right] \tag{20}$$

Finally, using the Stirling approximation,

$$S_E = A_E \left[(L+K)\, ln\,(L+K) - L\ln(L) - K\ln(K)\right]. \tag{21}$$

About this equation, it is essential to clarify that L and K must be dimensionless quantities. Therefore, they should not depend on the units in which they are measured. Hence, these quantities must be indexed, as done by Cobb-Douglas (1928). As a result, A_E, the economic Boltzmann constant has the corresponding units of the production function, i.e., monetary units. A_E is analogous to k_B in the thermodynamics equation (15) and consequently links production to the distribution of products after commercialization (Mimkes, 2006). Therefore, to calculate the production function based on the relationship with thermodynamics (equation 21), a fit of the constant A_E is required.

II. Methodology

To evaluate and compare the proposal to calculate the production function through economic entropy, as presented in the previous section, we use the data published by Cobb and Douglas (1928). These data include information on labor (L) and capital (K), as in equation (21), as well as production data for comparison with our proposal. Following the same procedure as these authors, the mentioned quantities are indexed in such a way that dimensionless quantities are obtained for use in equation (21).

In their procedure, the authors initially incorporate estimated annual additions to fixed capital in manufacturing and the total accumulated capital expressed in terms of costs and prices from 1880 for their capital variable (K), along with the probable average number of wage-earning employees in the manufacturing industry for their variable (L). For production (Q), the authors consider the growth of the physical output estimated by the index of the physical volume of production (Cobb and Douglas, 1928). Since the initial measurement scale for these three variables varies from one another, the values for the year 1899 are taken as a reference, and each of them is considered as a reference value assigned the value of 100 (each taken as 100% at this date). This is done to visualize the percentage growth or decrease of these quantities with respect to their references from the year 1899.

Since the initial measurement scales for both variables are different, they were indexed using the value at the initial time (1899) as a reference, with each case taken as 100. Similarly, for production (Q), the authors consider the growth of the physical output estimated by the index of the physical volume of production (Cobb and Douglas, 1928), which is already indexed to this year with a value of 100. In this way, the values of L, K, and Q all start at 100, and all subsequent values are indexed with respect to this value.

III. Results

First, as previously stated, it was necessary to revise the least squares estimation proposed by Cobb and Douglas (1928) in the context of robust statistics in accordance with the Gauss-Markov theorem, using the same data reported by the authors for L and K. Regression estimations and the corresponding tests were performed using the free software R. The regression results are shown in Table 2.

Table 2. *Least-Squares Estimation of the Data Reported by Cobb and Douglas (1928)*

Coefficients	Estimate	Std. Error	t value	$Pr(>\|t\|)$
$\log A$	-0.181	0.438	-0.414	0.683
α	0.81	0.146	5.532	1.73E-05***
β	0.23	0.064	3.596	0.0017**

Source: Author´s own calculations based on data from Cobb and Douglas (1928).

According to these results, the corresponding p-values for rejecting a null parameter value are near zero, indicating that the variables L and K explain the production behavior with strong statistical significance. Nevertheless, the hypothesis tests of the three parameters, $\log A$, α, and β, indicate that only the last two variables are statistically significant. The high value of 0.683 for the probability of rejecting the hypothesis that $\log A = 0$ implies that A is statistically equal to one. This contradicts the estimated value of $\log = -0.181$; $A = 0.834$. A similar analysis for the regression by Cobb-Douglas (1928) shows that only α and β are statistically significant, and again, $\log A$ is statistically equal to zero with a high probability, resulting in a value of 1.0 for A, as opposed to the Cobb-Douglas (1928) reported value of 1.01. Moreover, the reported estimations of 0.75 and 0.25 for α

and β, respectively, were not explained properly by Cobb and Douglas (1928). Finally, the corrected least squares model using the diagnostics required by the Gauss-Markov theorem arrives at $A = 1.0$, $\alpha = 0.81$, and $\beta = 0.23$.

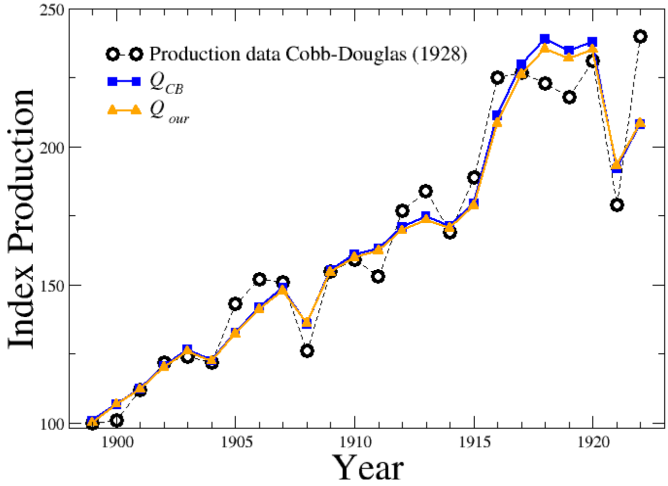
These exponents correspond to elasticities α and β of labor and capital, respectively, which are complementary, i.e., α and β sum to one and indicate constant scale returns in production. In the Cobb-Douglas case, the elasticity of L is three times that of K; that is, the marginal productivity of labor is three times greater than that of capital. Our results demonstrate that α and β are approximately complementary and that labor elasticity is approximately 3.5 times more productive than capital.

The Ordinary Least Squares (OLS) estimation by Cobb-Douglas (1928) shows, as one of its results, that the sum of α and β is complementary in the sense that their sum equals one. In economics, this model has been extensively used for calculating the production function. Thus, the complementary values of α and β are interpreted as constant scale returns, meaning that production will increase in the same proportion to the rise in L and K (Pindyck & Rubinfeld, 2017; Varian, 2014). Here, α represents the elasticity of output with respect to labor, and β represents the elasticity of production with respect to capital.

A comparative graphic view of both fits is presented in Figure 1, which depicts the reported production data (open black circles), the estimation made by Cobb-Douglas (1928) (orange circles), and our calculations (blue circles). The qualitative and quantitative behaviors of Cobb-Douglas and our fits are very similar compared with the production data.

Using the indexed data of labor and capital extracted from Cobb-Douglas's (1928) work, we compute production using the Stirling approximation equation (21) with $A_E = 1.0$. These results are compared with the production data reported by the same authors in Figure 2. Derived from the concepts of Richmond *et al.* (2013) and Mimkes (2010), the value $A_E = 1.0$ is assigned for the Boltzmann constant in their model since the authors assume that the production function computed by means of entropy does not require adjustable parameters. These results indicate the same qualitative behavior in both calculations, but the production values calculated by entropy with the

Figure 1. *Comparison of indexed production data extracted from Cobb-Douglas (1928)*



Notes: Production data obtained by Cobb Douglas (1928) (black circles), production estimated by Cobb Douglas, Q_{CB} equation (1) (blue squares), production obtained by our least squares estimation, Q_{our} (orange triangles) according to Table 2 results.
Source: Author's own calculations based on data from Cobb and Douglas (1928).

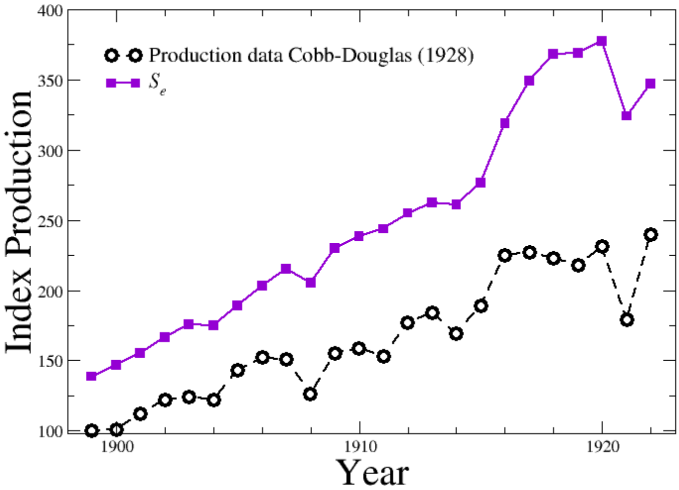
mentioned value of A_E are clearly overestimated.

These findings reveal that the constant A_E must have a value different from 1.0 so that the econophysics model (equation 21) can adequately explain production behavior. Based on some Gaussian regularity conditions avoiding extremal points known as influence (leverage-outliers), we can use the mean sample of deviations for estimating the constant A_E:

$$A_E = \frac{\sum_i Q_i / S_{Ei}}{t} \tag{22}$$

where Q_i is the indexed production data obtained from the original data, S_{Ei} represents the data calculated using equation (21), and t denotes the number of years considered. Thus, we consider a relative difference between the observed variable Q and Economic Entropy S_E. According to equation (22) and the data, $A_E = 0.6747$. Therefore, by substituting into equation (22), it is obtained that entropy describes a behavior more closely aligned with the

Figure 2. *CUS manufacturing sector production data extracted from Cobb-Douglas (1928)and production calculated by entropy,* S_E

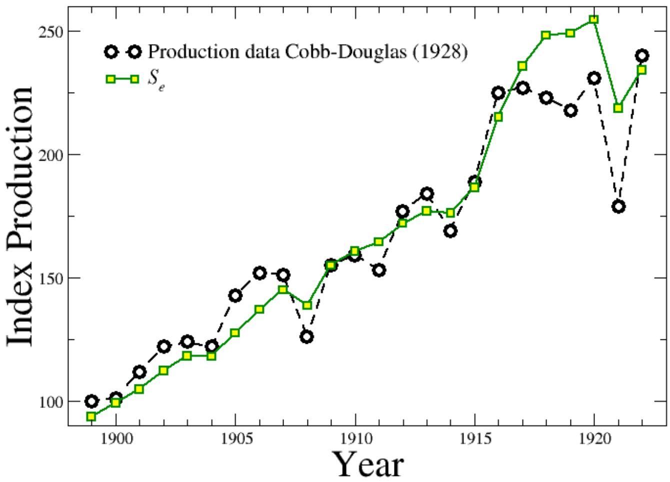


Notes: using equation (21) with $A = 1.0$.
Source: Author's own calculations based on data from Cobb and Douglas (1928) and equation (21) with $A = 1.0$.

original production data, as shown in Figure 3. This correspondence between the results indicates that adjusting the value of the Boltzmann constant is necessary to calculate the production function through entropy. A value of 1.0 results in an overestimated production, while an adjustment for these data using equation (22) yields a value that closely approximates the actual production.

It can be inferred that the behavior of economic entropy data provides a detailed description of the original production data, emphasizing the relevance of the constant A_E in the calculation of the production function through entropy. Thus, it is evident how a model based on principles of calculus and thermodynamics broadly captures the behavior of data in an economic system. This offers an alternative approach to addressing and modeling economic problems from the thermodynamics perspective, contrasting with models based on least-squares fits like the Cobb-Douglas

Figure 3. *US manufacturing sector output data from Cobb-Douglas (1928) and output calculated by entropy,* S_E



Note: Author's own calculations based on data from Cobb and Douglas (1928) and equation (21) with $A_E = \frac{\sum_i Q_i/S_{Ei}}{t}$

model. Although the latter may exhibit a more accurate estimation in some cases, it lacks a solid mathematical foundation.

Moreover, calculating production (or economic entropy) with a constant in the model becomes more akin to Boltzmann's theory, wherein k_B plays a fundamental role in linking entropy and disorder. In this case, A_E connects production to how the production process is carried out, which is related to order-disorder processes (Mimkes, 2006).

Although the work of Cobb and Douglas has indeed been a reference to production and consumer theory in economic theory, it has also been demonstrated that this function does not fit the data in all cases (see Fisher & Monz, 1992; Labini, 1995; McCombie, 1998 and Redondo, 2011). Furthermore, depending on the sector, its level of complexity may depend on many more factors than only two production factors— that is, the two adjustable elasticity parameters. Although our work aligns with the concepts reported in Mimkes' research (Mimkes *et al,* 2010 and Mimkes 2006), it suggests that the constant for economic entropy should be calculated. Then, the production results of an economic system can be described, as in the case studied by Cobb and Douglas (1928) for the manufacturing sector in the United States.

The discrepancies in Figure 3 can be explained by the extremal points that influence the performance of the mean given in equation (22). This fact will be the subject of future research.

Conclusions

The consolidation of physics's contributions to economics over the course of several decades has facilitated the emergence of a novel field of study known as econophysics. In this context, several applications have focused on the use of entropy for explaining or estimating economic variables. These include the contributions of Richmond *et al.* (2013), Mimkes (2006) and Mimkes *et al.* (2010), wherein several of the physical quantities can be related to essential quantities in the economy, including entropy and the production function. In these proposals, no direct comparisons have been made in production processes other than the optimization problems of two types of employees. In this regard, our methodology demonstrates a way to calculate the production function that is expressly dependent on labor and capital and is in perfect agreement with the results reported by Cobb and Douglas (1928). Therefore, our methodology allows direct comparisons with the traditional method of calculating the production function. Furthermore, this proposal is based on the relationship between economic problems and many-particle problems addressed in thermodynamics, unlike the traditional production function obtained as the least-squares fit.

Additionally, the model demonstrates the importance of calculating the economic Boltzmann constant, as it renders the production results comparable to those found in the economic problem. This contrasts with the assertions made by Mimkes (2006), Mimkes (2019) and Mimkes *et al.* (2010), who assume that this Boltzmann constant must have the value of unity. In our calculations, this constant takes a value of 0.6747 in the case of production data for the US manufacturing sector, as reported in Cobb-Douglas (1928) research. This demonstrates how a model based on calculus and thermodynamic premises manages to accurately describe the behavior of the data within an economic system. Moreover, this approach illustrates a method for describing and modeling economic problems using thermodynamics, unlike models based on least-squares adjustments, which, while providing accurate estimations in some cases, lack a mathematical basis to support them.

The fit between entropy and production can be enhanced by certain insights involving robust estimators of deviations that take into account influential points and the underlying error distributions. This will be investigated in a future study.

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Conflicts of Interest

The authors have no conflicts of interest to disclose, and there are no financial conflicts of interest related to the paper.

Data Availability Statement

The data that support the findings of this study are available on request from the corresponding author.

Ethics Statement

This research article did not work with a person or groups of persons to generate the data used in the methodology; therefore, it did not require the endorsement of an Ethics Committee for its realization.

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