Analysis of Students' Mathematical Errors as a Means to Promote Future Primary School Teachers' Diagnostic Competence

Análisis de los errores matemáticos de los estudiantes como medio para promover la competencia diagnóstica en los futuros profesores de primaria

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Abstract

The conceptualization of teachers' professional competencies has evolved in the last decades. The specification of the types of knowledge that teachers require, the inclusion of affective and motivational aspects and the acknowledgments of the processes that connect these dispositions to the behavior of teachers in the classroom are considered especially important. In particular, teachers' diagnostic competence has been regarded as crucial for successful teaching because it allows teachers to understand students' thinking and make a corresponding plan to promote learning. Errors have been recognized as a valuable source of information about students' thinking and therefore, teachers' diagnostic competence in error situation is the focus of this study. This article shares the design of a university course and the theoretical basis aimed at developing pre-service primary school teachers' diagnostic competence in error situations within their initial teacher education programs. Its implementation in Chilean universities suggests a valuable opportunity for future teachers to learn and discuss about mathematics and its teaching and learning.

Keywords: Teacher Competencies; Diagnostic Competence; Error Analysis; Mathematics Teachers' Competencies

Resumen

La conceptualización de las competencias profesionales de los maestros ha evolucionado en las últimas décadas. De especial interés resultan las especificaciones de los tipos de conocimiento que requieren los docentes, la incorporación de aspectos motivacionales y afectivos y el reconocimiento de los procesos que conectan estas disposiciones con el comportamiento de los maestros en el aula. En particular, la competencia diagnóstica de los docentes se ha considerado crucial para una enseñanza exitosa, ya que les permite comprender el pensamiento de los estudiantes y, en consecuencia, planificar para promover el aprendizaje. Los errores se reconocen como una valiosa fuente de información sobre el pensamiento de los estudiantes y, por lo tanto, el enfoque de este estudio es la competencia diagnóstica de los maestros en situaciones de error. Este artículo comparte el diseño y la fundamentación teórica de un curso universitario destinado a desarrollar la competencia diagnóstica en situaciones de error de los futuros maestros de primaria, dentro de sus programas de formación inicial docente. Su aplicación en universidades chilenas sugiere que es una valiosa oportunidad para que los futuros maestros aprendan y discutan sobre las matemáticas y su enseñanza y aprendizaje.

Palabras clave: competencias docentes; competencia diagnóstica; análisis de errores; competencias de docentes de matemática

Introduction

Teaching mathematics effectively in primary school classrooms poses several challenges for teachers. Under a student-centered paradigm, teachers need a set of professional competencies to plan and carry out lessons that consider the needs of all children and provide them with sufficient and suitable opportunities to learn. At the same time, teachers have to be able to support students individually and, therefore; they need to understand students' thinking. It is strongly called for in the discussion on noticing (Sherin, Jacobs & Philipp, 2011).

Professional competencies that teachers require to guide their students' learning process have been widely described and researched in the field of mathematics education (Shulman, 1986; Ball, Thames & Phelps, 2008; Kaiser, Blömeke, König, Busse, Döhrmann & Hoth, 2017). In particular, teachers' ability to understand students' thinking has been identified as crucial to promote differentiation and individualization among students. In other words, diagnostic competence allows teachers to comprehend students' ways of reasoning and adapt their teaching strategies accordingly to promote learning.

Errors occurring during the learning process are very often a rich source of information to interpret students' understanding about mathematical concepts and procedures (Radatz, 1979; McGuire, 2013; Brodie, 2014). Therefore, teachers should learn about errors, how to interpret and analyze them and begin to develop their diagnostic competence during their initial teacher education programs. This is, in turn, a challenge for teacher educators.

This paper describes a brief university course aimed at introducing future primary school teachers into the value of analyzing errors for the improvement of the learning situations and also, at building the foundations for the development of their diagnostic competence. The goal is that after the four sessions, preservice teachers may view errors as a useful source of information about students' mathematical understanding and have a tool for identifying, interpreting and deciding how to deal with student errors.

The course is at the core of a study aimed at investigating how future primary school teachers' diagnostic competence in error situations can be assessed and fostered within initial teacher education. This course was offered in four Chilean university settings. More than 130 pre-service primary teachers on their third or fourth year of university studies took part on it.

Theoretical Background

Teachers' Professional Competencies

The concept of competence has acquired significant relevance in the past decades in the fields of psychology and education, accompanied by rich discussions regarding its significance, interpretation and assessment. Competencies are considered to comprise more than pure knowledge and skills. These are also context-specific, they involve the ability to use cognitive, affective, motivational and social capabilities to act adequately in real context situations (Koeppen, Hartig, Klieme & Leutner, 2008; Weinert, 2001).

The study of the professional competencies of teachers was strongly influenced by the seminal work of Shulman (1986). In his famous contribution, he distinguished between subject-matter knowledge and general pedagogical knowledge. Subject-matter content knowledge refers to the body of knowledge of the domain to be taught in a broader way. It includes understanding the structure of the particular subject in a way that allows teachers "to explain why a particular proposition is deemed warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice" (p. 9). General pedagogical or curricular knowledge includes being familiar with a wide range of teaching programs, strategies and instructional materials to teach particular topics at certain grade levels, and awareness of the circumstances in which those programs, strategies and materials might be effective or rather not. Additionally, it covers knowledge about the topics taught before and after within the same subject, and in parallel in other subjects.

However, he pointed out that these two domains are not enough for effective teaching, what he called pedagogical content knowledge is also needed. Pedagogical content knowledge refers to "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). It comprises a broad range of representations, examples and explanations that may prove useful to promote students' learning, knowledge about aspects of a topic that make it easier or more difficult for students to grasp and knowledge about common preconceptions and misconceptions for a particular content and strategies that can be used to aid students in overcoming those difficulties.

With the aim of more precisely describing the content knowledge areas and clarify further how teachers are expected to understand the contents they teach, researchers from the University of Michigan analyzed the work done by primary school teachers within the project Mathematical Knowledge for Teaching (Ball, Thames & Phelps, 2008). Their results showed that "the mathematical demands of teaching are substantial. The mathematical knowledge needed for teaching is not less than that needed by other adults. In fact, knowledge for teaching must be detailed in ways unnecessary for everyday functioning" (p. 396). They called this type of professional knowledge "mathematical knowledge for teaching" and they described it theoretically by organizing it into several subcomponents, which can be classified into subject matter knowledge and pedagogical content knowledge, as it can be seen in Figure 1.

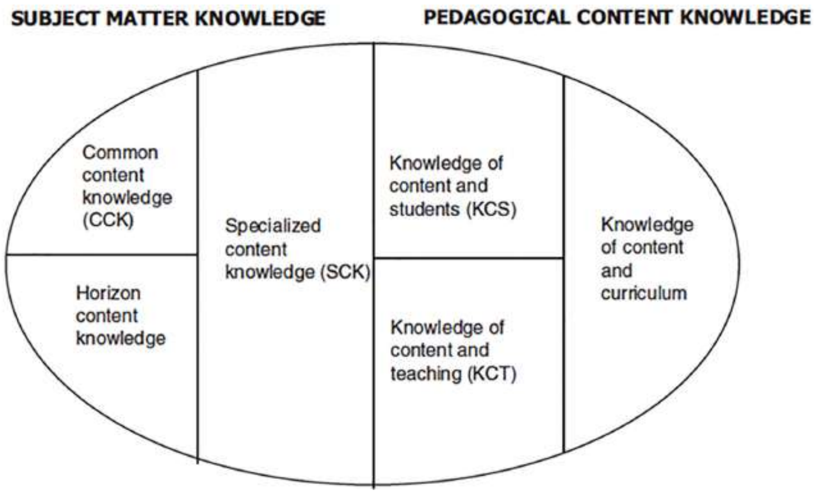


Figure 1. Mathematical knowledge for teaching model (Ball, Thames & Phelps, 2008, p. 403)

The three subcomponents included in the area of pedagogical content knowledge are knowledge of content and students, knowledge of content and teaching and knowledge of content and curriculum. The first, knowledge of content and students, covers the link between familiarity with students' ways of learning and the particular subject. This is, for example, the ability to anticipate issues that may be confusing for some students, the ways students may be reasoning about a particular topic or the answers they may give in a certain task. It can also include the ability to interpret and understand students' thinking and arguments that may be expressed in everyday language and sometimes even incomplete and knowledge about common errors or misconceptions that may arise during the learning of certain topics. Similarly, the second subcomponent, knowledge of content and teaching, includes the connection between knowing about mathematics and about teaching. For instance, when teachers have to select examples and representations to introduce a particular topic they have to weight their level of difficulty and their instructional advantages and disadvantages. In addition, during a lesson, they have to make decisions related to the convenience of deepening (or not) into a student's contribution, the need to clarify further an issue, to make a question or give a particular task to promote understanding or generate conflict to the reasoning of a student. The last and third subcomponent, curricular knowledge, comes from Shulman's (1986) categories. It was provisionally located by the research team in the pedagogical content knowledge area, but they left open the issue suggesting that it may also be a part of knowledge of content and teaching, run across different subcomponents or also constitute a separate domain (Hill et al., 2008; Ball et al., 2008).

The area of subject matter knowledge covers common content knowledge, specialized content knowledge and horizon content knowledge. The first refers to the "mathematical knowledge and skill used in settings other than teaching" (Ball et al., 2008, p.399) and it is justified because teachers obviously need to know mathematics itself if they are teaching it. They have to be able to distinguish between wrong and right answers and between accurate and inaccurate definitions. They also have to use concepts and notations correctly. On the contrary, specialized content knowledge is "the mathematical knowledge and skill unique to teaching" (p. 400) and thus "not typically needed for purposes other than teaching" (p. 400). It involves a special form of understanding mathematics beyond the contents being taught. Interpreting students' procedures to find error patterns or to decide if non-standard procedures are mathematically correct would constitute examples of this domain. The last subcomponent, horizon content knowledge, refers to "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). This would allow teachers to bear in mind what comes later so they can build solid mathematical foundations.

Also, on the basis by Shulman's (1986) categorization, but broadening the understanding of what teachers need to teach mathematics to include affective-motivational domains, the international comparative Teacher Education and Development Study in Mathematics (TEDS-M) developed the conceptual model illustrated in Figure 2 (Döhrmann, Kaiser & Blömeke, 2014). This framework is based on the concept of competence by Weinert (2001), so teachers' professional competencies include both cognitive and affective-motivational facets. On the cognitive side of the model, three knowledge components can be distinguished: mathematical content knowledge (MCK), mathematical pedagogical content knowledge (MPCK) and general pedagogical knowledge (GPK).

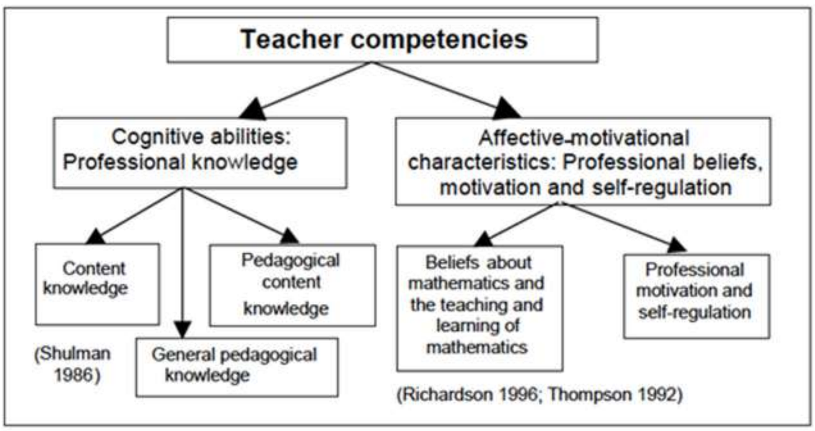


Figure 2. Conceptual model of teachers' professional competencies used in the TEDS-M study (Döhrmann et al., 2014, p. 435)

Mathematical content knowledge (MCK) is the knowledge of the discipline. Its facts, structure and principles are organized in the TEDS-M framework into four content domains, namely numbers, geometry, algebra and data and probability. Mathematics pedagogical content knowledge contains curricular knowledge, the knowledge and skills required to plan and select strategies for the teaching and learning of mathematics and the knowledge needed to put those strategies effectively into practice (Tatto et al., 2008). Mathematical pedagogical content knowledge (MPCK) includes abilities such as selecting appropriate teaching strategies, choosing assessment formats, representing and explaining mathematical ideas, understanding standard and non-standard methods to solve mathematical problems, predicting areas of students' difficulties, common errors and typical responses, evaluating students' mathematical thinking, generating fruitful questions and providing appropriate feedback (Döhrmann et al., 2014). The TEDS-M framework also considered general pedagogical knowledge as part of the cognitive abilities in their framework. However, this area was assessed only in a few participating countries (König et al., 2011).

In addition to the cognitive components, in line with a competence-based approach, the TEDS-M framework distinguished an affective-motivational dimension that included beliefs about the nature of mathematics and about the teaching and learning of mathematics and teachers' professional motivation and self-regulation (Tatto et al., 2008). Teachers' beliefs are recognized to play an important role on how they interpret classroom situations and make decisions (Schoenfeld, 2011). Motivation and metacognitive abilities such as self-regulation allow teachers to set, monitor and achieve their own objectives in order to overcome difficulties and develop professionally.

More recently, Blömeke, Gustafsson and Shavelson (2015) conceptualized teachers' competence as a continuum. They identified, amongst other discrepancies, two opposed positions in understanding competence that also led to differences at the methodological level. On one side, the analytical approach focuses on the complexity of the abilities involved in the conceptualization of competence and intends to divide it into cognitive and affective-motivational resources. On the other side, the holistic approach emphasizes the real-life part of the conceptualization, considers that cognitive and affect-motivational traits are constantly modified during performance and thus seeks to focus on behavior in context. By taking this discrepancy as a starting point and recognizing that both positions worked from assumptions that can be agreed upon, Blömeke et al. (2015) suggested to go "beyond dichotomies" and proposed a model to conceptualize competence as a continuum.

The authors' model, illustrated in Figure 3, considers cognitive and affective-motivational traits as resources that are available for a person to put into practice and, at the same time, recognizes the crucial role of observing how these are integrated into performance in real-context situations. They studied the process that connects disposition facets and integrates them into the observable performance and identified three skills that may act as a bridge in the process, namely perception, interpretation and decision-making. They further suggest that discrepancies between both positions should be avoided and "competence should be regarded as a process, a continuum with many steps in between" (p. 7). Moreover, they suggest that the model may serve as a tool for research on competence development to focus on the steps of the process in which resources are activated and mediated by perception, interpretation and decision-making skills to result in performance in real-life situations.

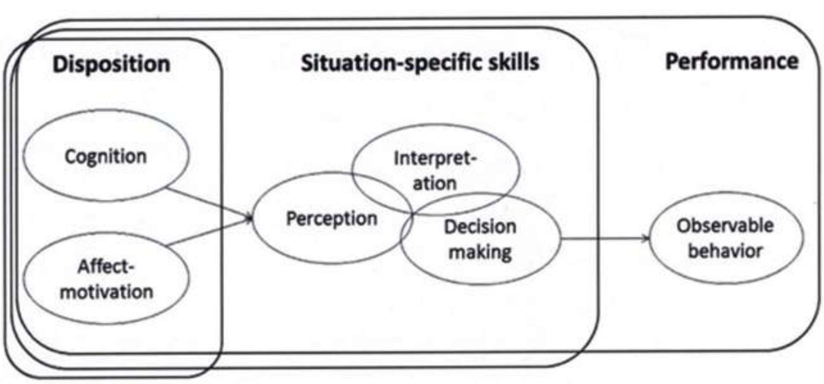


Figure 3. Model of competence as a continuum (Blömeke et al., 2015, p. 7)

Diagnostic Competence in Error Situations

Diagnostic competence is acknowledged as one special facet of teachers' professional competencies. Although the word "diagnosis" might have some clinical or medical connotations because one of the main tasks of physicians is diagnosing patients to make decisions about their treatment, teachers also perform diagnostic activities in their daily tasks. Teachers need to assess students' learning outcomes and processes in order to plan further teaching activities. They constantly identify and analyze both individual student's and whole-group's current levels of understanding to take them as a starting point to plan further instructional activities aimed at promoting learning. The body of resources needed to carry out these activities has been defined as *diagnostic competence.*

Some research approaches have been understanding teachers' diagnostic competence as the accuracy of their judgments in relation to students' achievements (Helmke & Schrader, 1987). Here, we focus on teachers' competence to gather information during class about their students' mathematical understanding, their difficulties and misconceptions, and make ongoing analyses that allow them to provide appropriate pedagogical responses, what has been called *situation-based diagnostic competence* (Hoth et al., 2016).

Prediger (2010) points out that diagnostic competence draws on four elements from both cognitive and affective-motivational domains. Not only knowledge about mathematics concepts and skills and mathematics learning are necessary to understand students' thinking, but also affective components, that include teachers' beliefs, curiosity about students' thinking and an interpretative attitude, play a crucial role in teachers' understanding of the underlying reasoning of students' thinking.

An important source of information for understanding students' thinking is the mathematical errors they make during the learning process. Errors, considered as "systematic, persistent and pervasive mistakes" (Brodie, 2014, p. 223), that students cannot identify and correct by themselves, provide teachers with valuable information about the flaws on students' mathematical reasoning. In fact, the value of the mathematical errors found in the work of students or identified during classroom interactions relies precisely in the evidence they provide for teachers about students' erroneous conceptualizations and about where students' knowledge and skills need further support, so appropriate pedagogical resources can be put in place (McGuire, 2013, Brodie, 2014, Radatz, 1979).

Errors usually make sense for the student, because they are the result of erroneous conceptualizations, they are anchored into cognitive structures built by the student. This implies that for overcoming the error, complex cognitive restructuring needs to take place (Brodie, 2014). Hence, it poses a significant challenge for teachers, who besides recognizing and analyzing the student error need to make pedagogical decisions and design instructional strategies that help the student recognize the incorrectness of their reasoning and reorganize their knowledge.

To conceptualize the complex process teachers undergo when they diagnose students' errors, Heinrichs and Kaiser

(2018) defined diagnostic competence in error situations as.

> The competence that is necessary to come to implicit judgements based on formative assessment in teaching situations by using informal or semi-formal methods. The goal of this process is to adapt behavior in the teaching situation by reacting to the student's error in order to help the student to overcome his/her misconception. (p. 81)

Taking as a starting point various models of teachers' diagnostic competence and error analysis, Heinrichs and Kaiser (2018) identified three common steps that were present in every model, namely perceiving or identifying, understanding or interpreting and finally deciding how to proceed. Based on this, they developed a model for future teachers' diagnostic competence in error situations, which also consists of three steps and is illustrated in Figure 4. In the first phase, teachers pay attention to students' work or to a particular classroom situation and identify or notice the error. The identification of the error is evidently essential to generate a pedagogical reaction to it. In the second step of the process, teachers look for causes of the error. A fruitful analysis of an error involves more than perceiving the error, it requires that teachers are able to look at specific characteristics of the error and interpret them in accordance with the particular learning situation. They also have to look at the type of error and, make hypotheses with the available information about possible underlying causes for that error in that specific situation. Finally, the third phase is dealing with the error. Considering the hypotheses about the sources of the error and knowledge about teaching and learning of mathematics, teachers plan a pedagogical strategy aimed at helping the student overcome their error and promoting further learning.

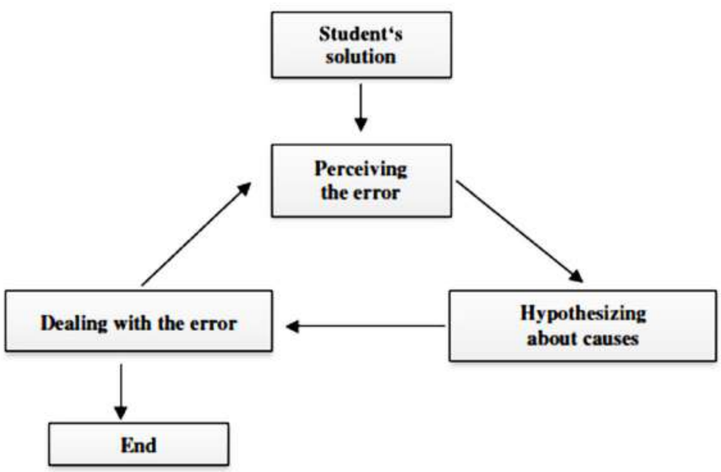


Figure 4. Model of the diagnostic process in error situations (Heinrichs & Kaiser, 2018, p. 84)

This model is also in line with the model of competence as a continuum of Blömeke et al. (2015) as well as the concept of noticing, which played a significant role in follow-up-studies of TEDS-M (cf. Kaiser et al., 2015). It acknowledges that teachers draw on some dispositions, including knowledge, beliefs, motivation and affective aspects to put in practice the situation-specific processes of perception, interpretation and decision-making that lead to their observable performance in diagnostic situations. It is evident how the perception phase is very similar to the error identification stage of Heinrichs and Kaiser's (2018) model, as it consists on the first contact, the acknowledgement of the situation. The interpretation part is also closely related to the second phase of the diagnostic process in error situations, which is generating hypotheses about the causes of the student's error, as it involves taking into account the available information to analyze it and try to better understand the teaching and learning situation, in this case the mathematical (erroneous) thinking of the student. In both models, the last process refers to making pedagogical decisions to foster learning; in particular it involves the strategies selected by the teacher that would support the student in overcoming his or her error.

Fostering Diagnostic Competence in Initial Teacher Education

If teachers are to consider students' current level of mathematical understanding as a starting point to foster further learning, it becomes evident that promoting teachers' diagnostic competence should be included as a relevant component in initial teacher education programs. Although teaching experience is a key in the development of such competence, it has been argued that it may be beneficial for future teachers to build a base of knowledge and skills by having initial experiences to develop these competencies, in which appropriate support and guidance are available (Cooper, 2009). Hence, teacher educators face the challenge of finding and providing such learning opportunities within pre-service teacher education.

The use of both written samples of students' work and videos showing them on task or classroom situations has been suggested by Jacobs and Philipp (2004) as a useful tool for triggering rich analyses and discussions about students thinking and thus facilitating the development of knowledge about mathematics, its teaching and learning. They highlight that the main value of including those samples of students' work into teacher preparation relies not in the work samples themselves, but in the discussions that they may generate if interesting and relevant questions are brought in by teacher educators and future teachers engage in productive discussions and analyses about mathematics, teaching and learning.

Also Blömeke et al. (2015) and Kaiser et al. (2015) point out the usefulness of videos in the context of competence assessment "using representative job situations so that the perception of real-life, that is unstructured situations, can be included" (Blömeke et al., 2015, p. 9). This desire to use assessment prompts as near to real classroom situations as possible can also be extended to learning opportunities aimed at developing teaching competencies. The use of videos should allow future teachers to situate themselves closer to a real situation and thus perceive the situation more holistically, trying to include various elements of the situation at hand.

The generation of productive discussions from the videos may prove challenging for teacher educators. Jacobs, Lamb and Philipp (2010) provide some prompts that may be helpful in guiding the process. First, they point out that teacher educators may need to provide directed support so pre-service teachers learn to shift the focus of the discussions from general issues to learners' understanding specifically, to recognize mathematically and pedagogically significant elements in students' explanations. They also suggest that some of pre-service teachers' difficulties in interpreting student' work may be a consequence of deficits on their knowledge about mathematics and mathematics teaching, thus they would require support to make sense of students' strategies and connect them appropriately to mathematical concepts. Also, the authors stress that, although future teachers' suggested pedagogical responses may vary widely, it is crucial for the proposals to be productive that teacher educators make sure they are based on children's understandings.

University Course for the Development of Pre-Service Teachers' Diagnostic Competence in Error Situations

Based on the discourse described above a brief university course of four 90-minute sessions was designed considering the relevant background literature. The teaching sequence aimed at promoting pre-service primary school teachers' diagnostic competence. During every session, participants were expected to engage in individual analysis of teaching and learning situations and in both small groups and whole-class discussions about students' work, their errors and understanding of mathematics. As most participating pre-service teachers did not have previous opportunities to learn about the potential of error analysis for understanding students' thinking, an additional goal was to sensitize them about the role errors can play in mathematics teaching and learning by using video clips and authentic student artefacts.

To approximate future teachers to real-life situations while staying in a university setting, students' work samples were presented both in paper copies and video clips from classroom situations. The discussions generated from these materials were supported both by the teacher educator directly and by sets of questions and prompts that led the focus towards students' learning processes and understanding of mathematics.

The model of diagnostic competence in error situations from Heinrichs and Kaiser (2018) was at the core of the design of the course structure and also presented to pre-service teachers as a three-step error analysis cycle that worked as a tool to facilitate their diagnostic thinking. During the four sessions, pre-service primary school teachers worked through the error analysis cycle several times. Various samples of students' work were used together with questions and prompts to emphasize particular steps of the cycle. Additionally, short texts from the literature relevant to the errors were handed out so discussions could be enriched with these perspectives.

Considering that poor mathematical or mathematics pedagogical knowledge may negatively affect pre-service teachers' ability to generate fruitful discussions about the errors, the selection of errors to be included into the sessions was narrowed down to examples within the area of number and operations and to contents covered in the Chilean school curriculum in the four primary grades. Additionally, errors were chosen from those reported in the specialized literature as occurring most often among primary school children. Moreover, it is worth acknowledging that the course was not intended to cover a wide variety of errors or to deeply study any particular errors, the goal was to foster pre-service primary school teachers' diagnostic competence.

Introductory Session

The goal of the course's first session was to sensitize pre-service teachers about the relevant role errors can play for teaching, as they provide useful information about students' mathematical thinking. An additional objective was to introduce the error analysis cycle that was used along all sessions.

In order to engage participants, the session started with a short video clip showing a third-grade student in a Chilean classroom working on a basic multiplication facts worksheet. The teacher approaches him as she identifies that he has written 5*0=5. On the first phase, preservice teachers were encouraged to take notes during the video and then briefly comment on it with a partner without any further guidance, so any ideas and analyzing strategies would arise.

The simplicity and commonness of this error were the main factors that determined the decision of including it into the session. Furthermore, it has been widely covered in the mathematics education literature. Although such an example may seem to be very simple, Van de Walle et al. (2014) have pointed out that despite it involves a simple procedure, students are often confused as they directly transfer rules they learned for addition. Padberg and Benz (2011) found the error n*0 = n to be the most common one leading to wrong results in written multiplication algorithms and suggest a number of other reasons for this error to occur.

After commenting their first discussions with the whole group and with the aim of explaining the role of errors in the teaching and learning process, participants received an extract of the article by Larrain (2016). Taking this as a base, the error analysis cycle was presented and explained in detail (Figure 5). To improve the usefulness of it, supporting questions for each phase were also provided (Table 1).

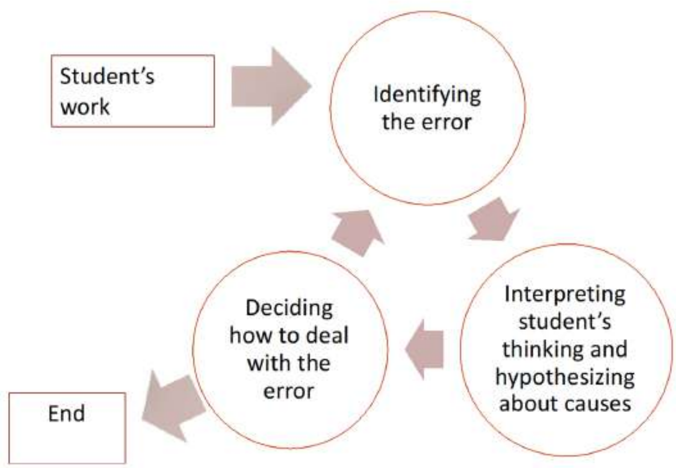


Figure 5. Error analysis cycle used within the course

Table 1

Supporting questions for each phase of the error analysis cycle

Phase	Questions
Identification	What is mathematically incorrect in the student's work? What is correct in the work of the student? What would be the right answer?

Interpretation	What could be the erroneous procedure used by the student? What could be the misconception underlying the error? What could have generated the erroneous thinking? What could be the causes for this error to occur in this particular situation?
Decision making	Do I need to collect additional information? How? What tasks, examples, questions or activities would help the student to recognize their error? What teaching strategy would be useful to help the student correct their error and promote their learning?

The same video clip with the error situation was shown again before pre-service teachers analyzed it using the error analysis cycle. In pairs, they systematized their discussion using the provided worksheet, which contained guiding questions for each step of the cycle. The questions asked teachers the following aspects: to specify what was mathematically incorrect in the student's work, to develop as many hypotheses as they could think of about why the student could be reasoning in that way, to answer some additional multiplication tasks as the student would do it by applying his misconception, to indicate questions and tasks they would give the student to indagate further into the student's thinking and to confirm their own hypotheses about the causes for the error based on earlier experiences and, finally, to briefly suggest some teaching strategies they would use to deal with the error.

To enrich their analyses, they formed groups of two pairs and received an extract of the text by Padberg and Benz (2011), describing this error from the perspective of the mathematics education field. The use of this text was crucial in widening their understanding of the error and the range of possible causes for it. To finalize the first session, their analyses were discussed with the whole class so different perspectives could be shared. In addition, the usefulness of the error analysis cycle was commented and a text from Selter and Spiegel (1997) about the ways in which children think differently was given to read before the second session. This text was intended to support the comprehension of the relevance of understanding students' thinking for promoting mathematics learning.

Session on Identifying and Hypothesizing about Causes of Students' Errors

The goal of the second session was to foster future teachers' competence to identify student errors and to hypothesize about causes for them. During this session, participants were involved in the analysis of written samples of students work and expected to describe the errors in detail and think of multiple possible causes for each particular error.

At the beginning of the session, the text by Selter and Spiegel (1997) was commented. At this stage, it was especially relevant for future teachers to acknowledge, in many cases for the first time, that children might think in a wide range of ways and use strategies very different from those future teachers themselves are familiar with. This highlighted also the need to develop flexible thinking and deeper mathematical understanding to be able to comprehend students' thinking.

Participants then received three samples of students' work on tasks related to the concept and the addition of fractions. Three different errors were provided. The first one, shown in Figure 6, was on the representation of fractions with the area model. Areas, in which each figure was divided were not equivalent in some of the models. This error has been indicated and analyzed in the mathematics education literature widely (Ashlock, 2010; Baroody & Hume, 1991).

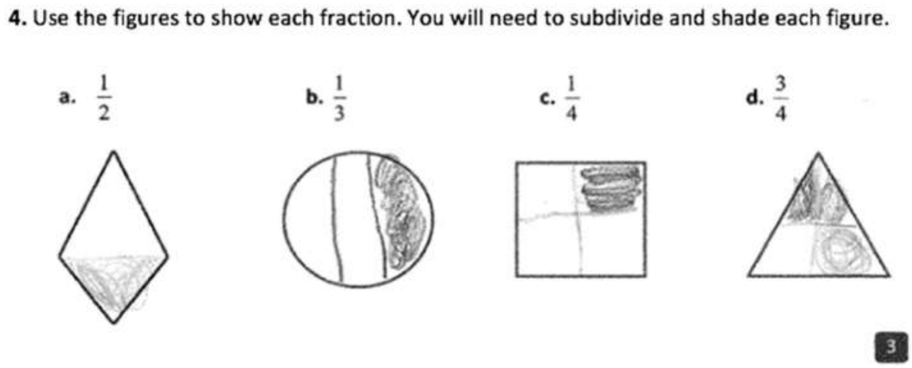


Figure 6. Sample of student's work with error on representation of fractions

The second error was adding the numerators and the denominators separately in the form, $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$ (Figure 7). This error has been recognized as a common error among students and documented by many authors, such as Rico (1995), Padberg (2002) and Ashlock (2010).

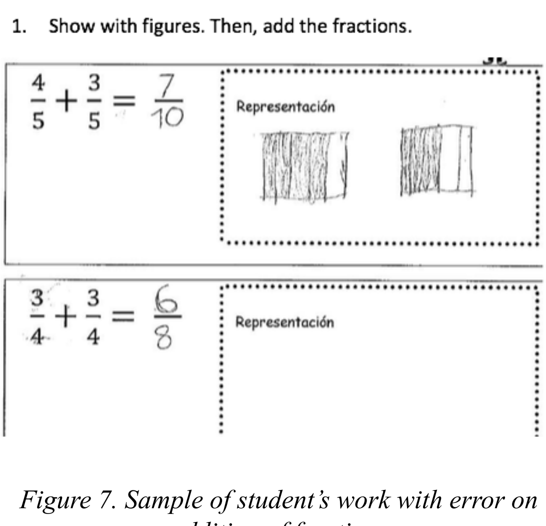


Figure 7. Sample of student's work with error on addition of fractions

The last error was also on the addition of fractions. In this case, the student had added the numerators and multiplied the denominators (Figure 8). Although this error is similar to the one in the previous case, it involves a different reasoning from the student (Ashlock, 2010).

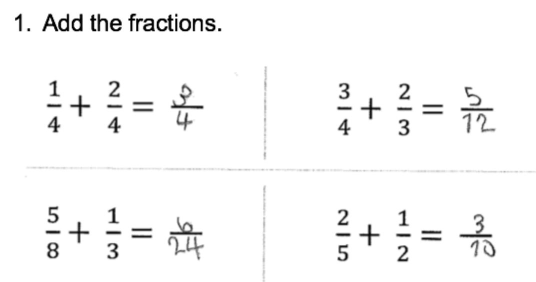


Figure 8. Sample of student's work with error on addition of fractions by multiplying denominators

The three error cases were accompanied by a worksheet to support the discussions and analyses future teachers should conduct. It provided questions for each case that aided the interpretation of the errors, it asked them to solve similar tasks applying the erroneous thinking of the student and offered guidance for the search of causes for each error. After discussing these analyses with the whole group, the session continued with a video that showed the student-teacher interaction related to the second error situation. The video, showing a familiar Chilean classroom situation, was important for complementing their analyses, as it provided additional and more detailed information about the student thinking, which led to a better understanding of the misconception involved and the points in which the student knowledge needed to be corrected or fostered. Future teachers worked first in pairs and then formed groups of four to enrich again their discussions, this time with the support from a brief translated text from Padberg (2002, p. 102-104) that contained a description of this error from an educational perspective. The session ended with a plenary discussion of their interpretations and hypotheses about causes for the error and a brainstorming activity about possible ways to work forward with the student.

Session on Decision-Making in Error Situations

The focus of the third session was the third phase of the cycle, i.e. the decision-making, and thus preservice teachers had to work through the complete cycle for each analyzed error. It was of special relevance at this step to promote that future teachers consider students' thinking and the current level of understanding when designing pedagogical strategies to deal with the error.

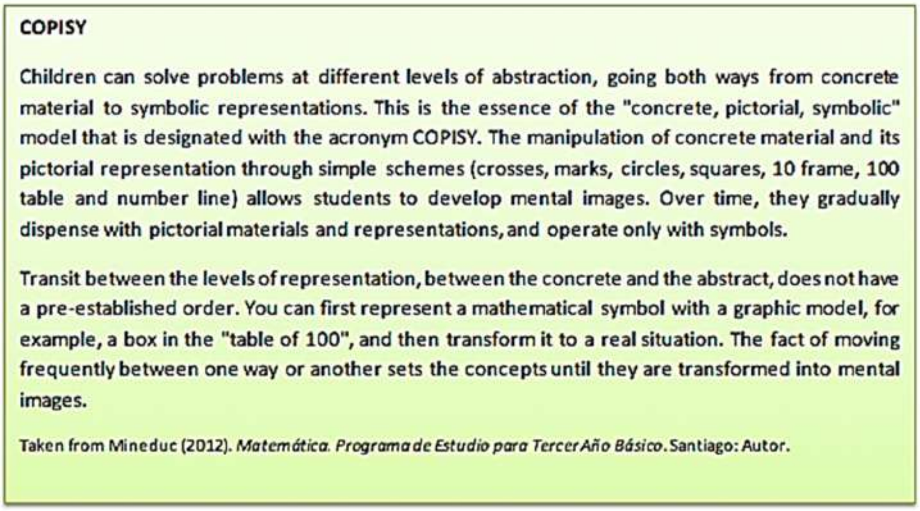
This session was divided into two main parts. In the first one, a video clip was used as a starting prompt. It showed the continuation of the student-teacher interaction of the error on the addition of fractions from the previous session (Figure 7). Participants were encouraged to comment on the strategies used by the teacher and provide some additional or alternative strategies they would have used to deal with the error in that particular situation.

As guiding support for designing their proposals, participants received a set of didactical principles for the teaching of mathematics taken from the Chilean primary school standards (Mineduc, 2012). A sample of them is shown in Figure 9. In addition, complementary information about the particular error was given by the translated text of Padberg (2002, p. 105-106) that continued the text they had received in the second session, providing some perspectives on the teaching of fractions. Finally, participants commented their strategies with the whole group. This activity served both purposes, i.e. to receive and provide feedback on the strategies each group had designed and to enrich their repertoires of pedagogical strategies ideas.

Mathematics with meaning

The teacher, from this perspective, should encourage students to make sense of the mathematical contents they learn and build their own meaning of mathematics to reach a deep understanding. In this sense, the teacher is expected to develop a pedagogical model that favors the understanding of mathematical concepts and not the mere repetition and mechanization of algorithms, definitions and formulas. For this, they must establish connections between concepts and mathematical skills. They are encouraged to carefully plan learning situations in which students can demonstrate their understandings and use a variety of concrete materials, to then move towards the use of images and iconic representations and progressively move towards a symbolic thinking that requires a higher level of abstraction.

Taken from Mineduc (2012). *Matemática. Programa de Estudio para Tercer Año Básico.* Santiago: Autor.



COPISY

Children can solve problems at different levels of abstraction, going both ways from concrete material to symbolic representations. This is the essence of the "concrete, pictorial, symbolic" model that is designated with the acronym COPISY. The manipulation of concrete material and its pictorial representation through simple schemes (crosses, marks, circles, squares, 10 frame, 100 table and number line) allows students to develop mental images. Over time, they gradually dispense with pictorial materials and representations, and operate only with symbols.

Transit between the levels of representation, between the concrete and the abstract, does not have a pre-established order. You can first represent a mathematical symbol with a graphic model, for example, a box in the "table of 100", and then transform it to a real situation. The fact of moving frequently between one way or another sets the concepts until they are transformed into mental images.

Taken from Mineduc (2012). *Matemática. Programa de Estudio para Tercer Año Básico*. Santiago: Autor.

Figure 9. Sample of didactical principles for the teaching of mathematics

In the second part, participants observed a video developed within the TEDS-FU study with an error on subtraction from a second-grade student, who revealed difficulties with place value issues. Because the focus of the video clip was not on the classroom context, but particularly in the students' reasoning and the student-teacher interaction, its European origin did not interfere with the task. Future teachers were encouraged to work through the complete error analysis cycle by using a worksheet that provided the questions associated to each of the three phases. They first had to identify the error, explain the erroneous strategy used by the student, specify the underlying concepts and procedures the student has not mastered as well as the areas in which he shows no difficulties. Taking this into account and the contextual information provided about the lesson and what had been learnt by the class, participants were asked to develop hypotheses about causes for the error in that situation. Lastly, they were requested to design a teaching strategy or a longer teaching sequence based on their hypotheses and on the information available on the student's mathematical thinking in order to help the student reconstruct the knowledge needed to overcome his error. As in previous activities, participants shared their analyses first in small groups and then with the whole group.

Closing Session

The last session provided pre-service teachers with the opportunity to apply the error analysis cycle to a particular error situation, then communicate their ideas and give and receive feedback. To do this, they formed groups of four and received a written copy of the work of a student containing an error. They were asked to analyze the error systematically, applying each phase of the error analysis cycle and using the corresponding questions as support. Their analyses were then displayed on a poster to be presented on a poster presentation activity later on in the session.

Six errors within the topic of operations with whole numbers under 1,000 were selected to be used in this activity. Half of the errors were related to subtraction and the other half to multiplication. The first subtraction error occurs when the standard written algorithm is meant to be used, regrouping is required twice within the same operation and the student "borrows" two directly from the hundreds place instead of regrouping one from the tens and one from the hundreds (exercises *c* and *d* in Figure 10). This error has been reported by Ashlock (2010) and Rico (1995) and acknowledged as revealing difficulties with the regrouping procedure in subtraction. The second sample of student work revealed an error on subtracting with zeros that has been documented in the literature by scholars such as Gerster (1982), Aslock (2010) and Padberg and Benz (2011). In this case, shown in Figure 11, the student solved n – 0 = 0 on each column with a zero in the subtrahend. The third subtraction error, illustrated in Figure 12, shows an error found by Lucchini, Cuadrado and Tapia (2006) that occurs by horizontal subtraction and is associated with difficulties in the understanding of basic ideas of place value.

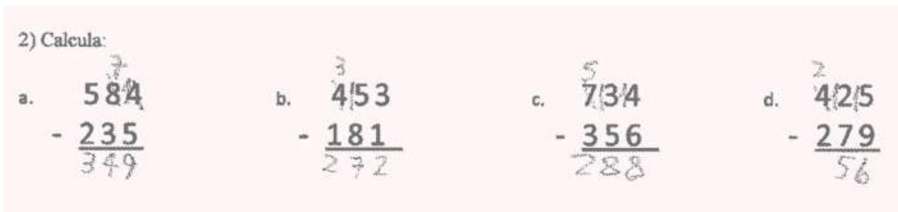


Figure 10. Error in subtraction algorithm with regrouping

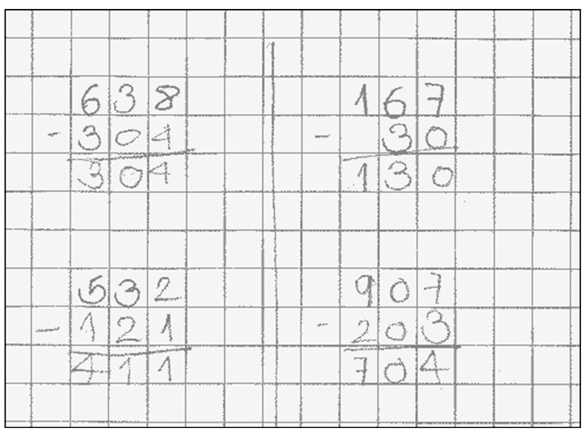


Figure 11. Error in subtractions with zero in the subtrahend

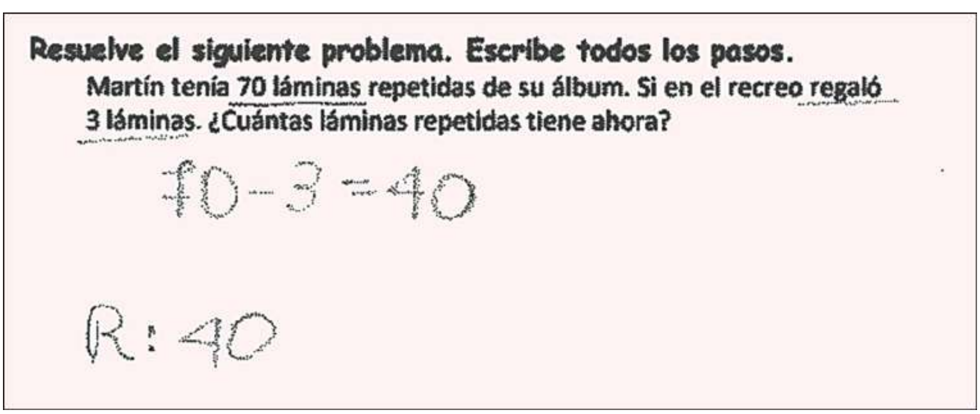


Figure 12. Error of place value in subtraction

The first case showing an error in multiplication, displayed in Figure 13, has also been identified by Ashlock (2010). It involves a wrong procedure for multiplying when the second factor is a two-digit number. The student multiplies the units of the second factor by the units of the first factor, notates the regrouping when there is one, and then uses the tens of the second factor to multiply the rest of digits on the first factor, leaving some partial products out and therefore leading to a wrong result.

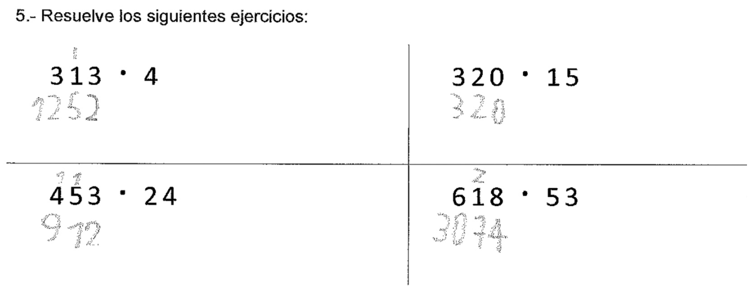


Figure 13. Error in multiplication algorithm by a two-digit multiplier

The second multiplication error (Figure 14) has also been discussed by Lucchini et al. (2006) and implies difficulties with place value as well. The student multiplies using the standard algorithm, starting by the units place, but instead of writing the result from right to left, writes down the result from left to right, which produces an inverted result.

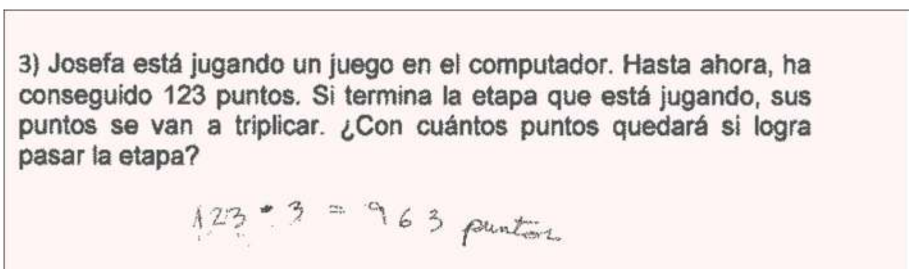


Figure 14. Error of place value in multiplication

The last error (Figure 15) occurs when a student intends to use decomposition to solve a multiplication task with two two-digit factors, but he fails to generate all of the partial products, registering only some of them. In other words, the student uses the distributive property, but multiplies units with units and tens with tens, and forgets the units by tens and tens by units partial products. This error has been reported by Padberg and Benz (2011).

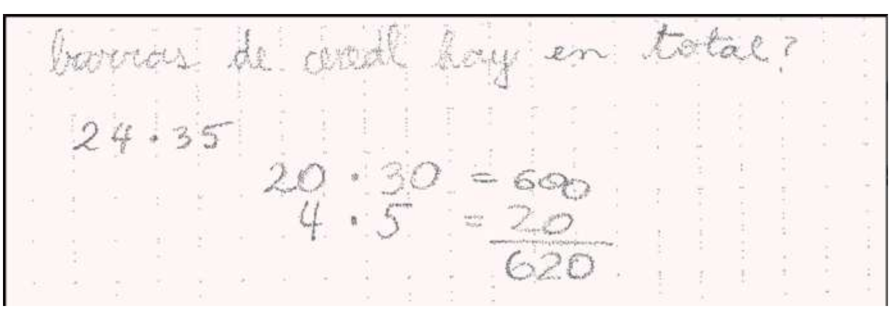


Figure 15. Error in using decomposition for multiplication

In the final phase of the session, participants presented their analyses to the group in a poster presentation. Two members of each group presented their poster to other visiting participants and the other two members went around visiting other posters. This allowed future teachers to look at various error analyses, evaluate and give their own views about other errors and the analyses made by other groups. It also gave them the opportunity to communicate and justify the analyses they made as a group. After some time, group members changed roles so everyone had the chance to be at their own poster and to visit others as well.

As a closure activity for the session and for the course, some of the errors of the poster presentation were commented, the usefulness of the error analysis cycle was discussed and pre-service teachers formulated questions or concerns that remained open until then.

Discussion and Outlook

The design of a teaching sequence aimed at developing professional competencies in future teachers presents great challenges for teacher educators. Within the situated approach of Blömeke et al. (2015), it is necessary to select activities that give participants the opportunity to place themselves in a context close to school reality, activate knowledge and put them at the service of the analysis and solution of a real problem. In addition, strategies are required that involve students with the proposed tasks, that encourage them to make conjectures, establish relationships between the situations presented and their knowledge and experiences and to get involved in group discussions. Likewise, learning opportunities such as the one described in this article implies continuous guidance, giving feedback at each activity and the proposals made by pre-service teachers.

In this sense, this course is distinct from other courses in didactics of mathematics in which some common student errors are included as content. The fundamental difference lies in the orientation with which these errors are addressed. In this course, it was not intended to discuss any particular error, but rather the value of errors as a source of information for the teacher; furthermore, the course should allow participants to learn to interpret student errors. In other words, as one participant said at the end of the course: "*here we went through a deeper process, not only knowing what the error is, but why [the student] is wrong about that"* (Anita, translated from Spanish by M.L.).

Especially useful were the written samples of student work and the videos used in the sessions as prompts in the sense suggested by Jacobs and Philipp (2004). These learning materials allowed future teachers to give importance to what they were going to learn because it linked the error situations to what would be a possible scenario of their professional work. It also gave them clues that allowed them to better understand the situation and thus provide better pedagogical answers to the particular situation. For instance, a future teacher put it this way:

> *"When watching a video, it's as if it happens to you too. As one is involved in the problem, listen to what the student says, which is different from someone telling you that or if it was written, listen to it with her words ... I perceive it differently when I see it in the video, with the tone what the student said, how she said it, is different*" (María, translated from Spanish by M.L).

As suggested by Jacobs et al. (2004), a point of great value in the course was the discussions generated from the error-cases provided. One participant highlighted the usefulness of putting into words, processes that are sometimes done intuitively: "*What helped me most were the discussions, verbalize what one thinks using mathematical language, share different visions to arrive at a better understanding of the error and better thought strategies. I had not thought of some techniques that came out from the group.*"(Anita). In addition, the enrichment that occurs in the exchange of ideas stands out. Future teachers expand their repertoire of strategies to deal with students' errors and make their thinking more flexible about the causes of such errors.

The course design addressed the need to sensitize preservice teachers to the role that errors can play in the teaching-learning process, as they provide relevant information about students' mathematical thinking and the areas in which they show difficulties and require support. Participants learned to react to the error not simply by pointing out the error to the pupil and then indicating the correct procedure or concept, but using the error analysis cycle based on Heinrichs and Kaiser's (2018) model as a tool that allows them to design a teaching strategy that takes as a starting point what the student knows and how s/he is reasoning mathematically. One participant said: *"with the cycle it is easier to think of a more elaborate strategy for the child, thinking about the individual, not just in general. You indagate, investigate what happens to the child and as you do, you apply a strategy and if it does not work, you start again. It serves to have an order, more tools to teach the child."* (Josefina, translated from Spanish by M.L).

Even more, participants indicated that this type of activities should be carried out more frequently, also for other areas of mathematics and in relation to other subjects. Valeria said: *"you could include more errors, but not only in numbers and operations, but errors in geometry ..., in the four areas of mathematics. [...] I find it very useful and practical [the course] and should also be done in the other subject areas: Language, history, science ... and as essential"* (translated from Spanish by M.L.). Thus, and in line with what Cooper (2009) suggested, it can be expected that this first step in the development of the diagnostic competence of future primary school teachers can be enriched with its application in different contexts.

Overall, the intervention study pointed out the necessity to include already in initial teacher education courses about diagnostic competence with a specific focus on students' errors. The usage of authentic student's solutions – so-called artefacts – and video clips with staged, but still authentic student-teacher-interactions allowed future teachers to gain semi-practical experiences, how to deal with students' errors in classrooms. These experiences were evaluated as extremely prolific by the participants and should be included in regular courses in mathematics education not only in Chile, but also at an international basis.

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